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TESTING PROCESS CAPABILITY USING THE INDEX $C_{\mathbf{pmk}}$ WITH AN APPLICATION

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Numerous process capability indices, including C_p, C_{pk}, C_{pm}, and C_{pmk}, have been proposed to provide measures on process potential and performance. Procedures using the estimators of C_p, C_{pk}, and C_{pm} have been proposed for the practitioners to use in judging whether a process meets the capability requirement. In this paper, based on the theory of testing hypothesis, we develop a step-by-step procedure, using estimator of C_{pmk} for the practitioners to use in making decisions. Then, the proposed procedure is applied to an audio-speaker driver manufacturing process, to demonstrate how we may apply the procedure to actual data collected in the factory.

Keywords: Process Capability Index; Testing Hypothesis; Critical Value; p-Value.

1. Introduction

Process capability indices (PCIs) have been widely used in the manufacturing industry, to provide a numerical measure on whether a process is capable of producing items meeting the quality requirement preset in the factory. Numerous capability indices have been proposed to measure process potential and performance. Examples include the two most commonly used indices, C_p and C_{pk} discussed in Kane,[1] and the two more-advanced indices C_{pm} and C_{pmk} developed by Chan *et al.*[2] and Pearn *et al.*[3] Those four PCIs have been defined explicitly as:

$$C_p = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sigma}\,, \qquad C_{pk} = \min\left\{\frac{\mathrm{USL} - \mu}{3\sigma}, \frac{\mu - \mathrm{LSL}}{3\sigma}\right\}\,,$$

$$C_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}}\,, \qquad C_{pmk} = \min\left\{\frac{\text{USL} - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - \text{LSL}}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\},$$

where USL is the upper specification limit, LSL is the lower specification limit, μ is the process mean, σ is the process standard deviation, and T is the target value. The index C_p only considers the process variability σ thus provides no sensitivity on process departure at all. The index C_{pk} takes the process mean into consideration but it can fail to distinguish between on-target processes from off-target processes (Pearn *et al.*[3]). The index C_{pm} takes the proximity of process mean from the target value into account, and is more sensitive to process departure than C_p and C_{pk}. The index C_{pmk} is constructed by combining the modifications to C_p that produced C_{pk} and C_{pm}. The ranking of the four basic indices, in terms of sensitivity to the departure of process mean from the target value, from the most sensitive one to the least sensitive are (1) C_{pmk}, (2) C_{pm}, (3) C_{pk}, and (4) C_p.

We note that C_{pk} is a yield-based index, and C_{pm} is a loss-based index. A process satisfying the quality condition "$C_{pk} \geq C$" may not satisfy the quality condition "$C_{pm} \geq C$". On the other hand, a process satisfying the quality condition "$C_{pm} \geq C$" may not satisfy the quality condition "$C_{pk} \geq C$" either. But, a process does satisfy both quality conditions "$C_{pk} \geq C$" and "$C_{pm} \geq C$" if the process satisfies the quality condition "$C_{pmk} \geq C$" since $C_{pmk} \leq C_{pk}$ and $C_{pmk} \leq C_{pm}$. According to today's modern quality improvement theory, reduction of process loss (variation from the target) is as important as increasing the process yield (meeting the specifications). While the C_{pk} remains the more popular and widely used index, the index C_{pmk} is considered to be the most useful index to date for processes with two-sided specification limits.

Since sample data must be collected in order to calculate the index value, then a great degree of uncertainty may be introduced into capability assessments due to sampling errors. Currently, most practitioners simply look at the value of the index calculated from the given sample and then make a conclusion on whether the given process is capable or not. This approach is intuitively reasonable but not reliable because sampling errors are ignored. Statistical theories about the index C_{pmk} have been provided extensively, but no step-by-step procedures are proposed for the practitioners in making decisions.

Cheng[4] and Cheng[5] have developed procedures using estimators of C_p and C_{pm} for the practitioners to use in judging whether a process meets the capability requirement. Pearn and Chen[6] used a proposed estimator of C_{pk} to develop a procedure similar to those described in Cheng[4] and Cheng,[5] so that the decisions made in assessing process capability is more reliable. But, no procedure for C_{pmk} was given because difficulties were encountered in calculating the sampling distribution of the estimator of C_{pmk}. In this paper, we first develop a procedure using a natural estimator of C_{pmk} proposed by Pearn *et al.*[3] Then, the procedure is applied to an audio-speaker driver manufacturing process to illustrate how we may apply the procedure to actual data collected in the factory.

2. Calculations of the C_{pmk} Value

To calculate process capability, sample data must be collected to obtain the C_{pmk} value. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normally distributed process with mean μ and variance σ^2 measuring the characteristic under investigation. Utilizing the identity $\min(a,b) = (a+b)/2 - |a-b|/2$, the index C_{pmk} can be alternatively written as:

$$C_{pmk} = \frac{d - |\mu - m|}{3\sqrt{\sigma^2 + (\mu - T)^2}},$$

where $d = (\text{USL} - \text{LSL})/2$ and $m = (\text{USL} + \text{LSL})/2$. Pearn *et al.*[3] considered a natural estimator of C_{pmk} can be defined as:

$$\hat{C}_{pmk} = \frac{d - |\bar{X} - m|}{3\sqrt{S_n^2 + (\bar{X} - T)^2}},$$

where $\bar{X} = \Sigma_{i=1}^n X_i/n$ and $S_n^2 = \Sigma_{i=1}^n (X_i - \bar{X})^2/n$ are the maximum likelihood estimators (MLEs) of μ and σ^2 which may be obtained from a stable process. We note that $S_n^2 + (\bar{X} - T)^2 = \Sigma_{i=1}^n (X_i - T)^2/n$ in the denominator of $\hat{C}_{pmk}$ is the uniformly minimum variance unbiased estimator (UMVUE) of $\sigma^2 + (\mu - T)^2 = E[(X - T)^2]$ in the denominator of C_{pmk}. For processes with target setting on the middle of the specification limits ($T = m$), which are fairly common situations, the natural estimator $\hat{C}_{pmk}$ can be rewritten as:

$$\hat{C}_{pmk} = \frac{D - \sqrt{W}}{3\sqrt{Y + W}},$$

where $D = n^{1/2}d/\sigma$, $Y = nS_n^2/\sigma^2$ and $W = n(\bar{X} - T)^2/\sigma^2$. Under the assumption of normality, the probability density function, the rth moment, and the first two moments as well as the variance of $\hat{C}_{pmk}$ can be obtained (see Pearn *et al.*[3]). We note that the estimator, $\hat{C}_{pmk}$, is biased. But, Chen and Hsu[7] showed that the estimator $\hat{C}_{pmk}$ is consistent, and asymptotically unbiased. Pearn *et al.*[3] investigated the moments of $\hat{C}_{pmk}$. Pearn *et al.*[3] also calculated the expected values and variances of $\hat{C}_{pmk}$ for $d/\sigma = 2(1)6$, $|\mu - T|/\sigma = 0.0(0.5)2.0$, and sample size $n = 10(10)50$. The first two moments of $\hat{C}_{pmk}$ can be expressed as the following, where $\lambda = n(\mu - T)^2/\sigma^2$:

$$E(\hat{C}_{pmk}) = \frac{e^{-\lambda/2}}{3}\sum_{j=0}^{\infty}\frac{(\lambda/2)^j}{j!}\left[\frac{d\sqrt{n}}{\sigma\sqrt{2}}\cdot\frac{\Gamma\left(\frac{n-1}{2}+j\right)}{\Gamma\left(\frac{n}{2}+j\right)} - \frac{j!\,\Gamma\left(\frac{n}{2}+j\right)}{\Gamma\left(\frac{1}{2}+j\right)\Gamma\left(\frac{n+1}{2}+j\right)}\right],$$

$$E(\hat{C}_{pmk}^2) = \frac{e^{-\lambda/2}}{9}\sum_{j=0}^{\infty}\frac{(\lambda/2)^j}{j!}\left[\frac{d^2 n}{\sigma^2}\cdot\frac{1}{n+2j-2} - \frac{d\sqrt{2n}}{\sigma}\cdot\frac{j!}{\Gamma\left(\frac{1}{2}+j\right)}\cdot\frac{2}{n+2j-1} + \frac{2j+1}{n+2j}\right].$$

Therefore, the bias and the MSE of $\hat{C}_{pmk}$ can be expressed as:

$$\text{Bias}(\hat{C}_{pmk}) = E(\hat{C}_{pmk}) - C_{pmk}\,,$$

$$\text{MSE}(\hat{C}_{pmk}) = \text{Var}(\hat{C}_{pmk}) + [\text{Bias}(\hat{C}_{pmk})]^2\,,$$

where $\text{Var}(\hat{C}_{pmk}) = E(\hat{C}^2_{pmk}) - E^2(\hat{C}_{pmk})$. Table 1 displays the comparison between the expected value of $\hat{C}_{pmk}$ and the actual value of C_{pmk} for $n = 50$ and various values $d/\sigma = 2(1)6$ and $|\mu - T|/\sigma = 0.0(0.5)2.0$. As $|\mu - T|/\sigma$ increases, both values of $E(\hat{C}_{pmk})$ and C_{pmk} decrease for fixed d/σ. We note that the difference between $E(\hat{C}_{pmk})$ and the value of C_{pmk} is the bias of $\hat{C}_{pmk}$. Furthermore, as $|\mu - T|/\sigma$ increases, we note that the variance of $\hat{C}_{pmk}$ also decreases for fixed sample size n and d/σ (see Pearn *et al.*[3]).

Table 2 displays the bias and the MSE of $\hat{C}_{pmk}$ for various values $d/\sigma = 2(1)6$, $|\mu - T|/\sigma = 0.0(0.5)2.0$, and sample sizes $n = 10(10)50$ in accordance with Pearn *et al.*[3] The results in Table 2 indicate that as d/σ increases, both the bias and the MSE increase for fixed sample size n and $|\mu - T|/\sigma$. On the other hand, as sample size n increases, both the bias and the MSE decrease for fixed d/σ and $|\mu - T|/\sigma$.

Table 1. Comparison between $E(\hat{C}_{pmk})$ and the value of C_{pmk} for $n = 50$, $d/\sigma = 2(1)6$ and $|\mu - T|/\sigma = 0.0(0.5)2.0$.

d/σ	$E(\hat{C}_{pmk})$ for $n = 50$					Values of C_{pmk}				
	$\lvert\mu - T\rvert/\sigma$					$\lvert\mu - T\rvert/\sigma$				
	0.0	0.5	1.0	1.5	2.0	0.0	0.5	1.0	1.5	2.0
2	0.6391	0.4562	0.2408	0.0949	0.0012	0.6667	0.4472	0.2357	0.0925	0
3	0.9775	0.7588	0.4792	0.2813	0.1511	1.0000	0.7454	0.4714	0.2774	0.1491
4	1.3160	1.0613	0.7177	0.4677	0.3010	1.3333	1.0435	0.7071	0.4623	0.2981
5	1.6544	1.3638	0.9561	0.6540	0.4509	1.6667	1.3416	0.9428	0.6472	0.4472
6	1.9928	1.6664	1.1945	0.8404	0.6008	2.0000	1.6398	1.1785	0.8321	0.5963

We note that $\hat{C}_{pmk}$ is a biased estimator. The results in Table 2 indicate that the bias of $\hat{C}_{pmk}$ is positive when $\mu \neq T$, i.e., C_{pmk} is generally overestimated by $\hat{C}_{pmk}$. When $\mu \neq T$, the results in Table 2 also indicate that both the bias and MSE of $\hat{C}_{pmk}$ decrease in $|\mu - T|/\sigma$ for fixed sample size n and d/σ, i.e., both the bias and MSE of $\hat{C}_{pmk}$ decrease as the value of $|\mu - T|/\sigma$ increases.

Constable and Hobbs[8] have found that, even if the underlying distribution of the data is normal, small samples do not provide acceptable estimates of the mean and standard deviation of the process. Proper sample sizes for capability estimation are extremely important. The smaller the sample, the higher $\hat{C}_{pmk}$ must be to demonstrate true process capability. However, remember that when sample size n

Table 2. The bias and MSE of $\hat{C}_{pmk}$.

$\frac{\lvert\mu - T\rvert}{\sigma}$	0.0		0.5		1.0		1.5		2.0	
d/σ	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
$n = 10$										
2	−0.0304	0.0382	0.0440	0.0388	0.0284	0.0189	0.0135	0.0068	0.0065	0.0026
3	−0.0025	0.0792	0.0681	0.0767	0.0432	0.0369	0.0214	0.0135	0.0108	0.0053
4	0.0254	0.1386	0.0921	0.1286	0.0581	0.0615	0.0293	0.0228	0.0151	0.0091
5	0.0533	0.2164	0.1162	0.1947	0.0730	0.0925	0.0372	0.0344	0.0193	0.0138
6	0.0812	0.3125	0.1403	0.2748	0.0878	0.1301	0.0451	0.0486	0.0236	0.0195
$n = 20$										
2	−0.0338	0.0166	0.0227	0.0171	0.0133	0.0078	0.0064	0.0030	0.0031	0.0012
3	−0.0207	0.0328	0.0340	0.0323	0.0203	0.0151	0.0102	0.0058	0.0052	0.0024
4	−0.0075	0.0561	0.0454	0.0529	0.0273	0.0249	0.0139	0.0097	0.0073	0.0041
5	0.0057	0.0865	0.0567	0.0790	0.0343	0.0373	0.0177	0.0147	0.0093	0.0062
6	0.0189	0.1240	0.0681	0.1105	0.0413	0.0523	0.0215	0.0207	0.0114	0.0087
$n = 30$										
2	−0.0317	0.0107	0.0152	0.0109	0.0087	0.0049	0.0042	0.0019	0.0020	0.0008
3	−0.0231	0.0208	0.0226	0.0203	0.0133	0.0094	0.0067	0.0037	0.0034	0.0016
4	−0.0144	0.0352	0.0300	0.0330	0.0179	0.0155	0.0091	0.0062	0.0048	0.0026
5	−0.0058	0.0540	0.0374	0.0491	0.0224	0.0232	0.0116	0.0093	0.0061	0.0040
6	0.0028	0.0771	0.0449	0.0685	0.0270	0.0325	0.0141	0.0131	0.0075	0.0056
$n = 40$										
2	−0.0295	0.0080	0.0113	0.0080	0.0065	0.0036	0.0031	0.0014	0.0015	0.0006
3	−0.0231	0.0153	0.0168	0.0147	0.0099	0.0068	0.0050	0.0027	0.0025	0.0012
4	−0.0166	0.0257	0.0224	0.0239	0.0133	0.0113	0.0068	0.0045	0.0036	0.0019
5	−0.0102	0.0393	0.0279	0.0355	0.0167	0.0168	0.0086	0.0068	0.0046	0.0029
6	−0.0038	0.0560	0.0334	0.0495	0.0201	0.0236	0.0105	0.0096	0.0056	0.0041
$n = 50$										
2	−0.0276	0.0064	0.0090	0.0063	0.0051	0.0028	0.0025	0.0011	0.0012	0.0005
3	−0.0225	0.0121	0.0134	0.0116	0.0078	0.0054	0.0039	0.0021	0.0020	0.0009
4	−0.0174	0.0203	0.0178	0.0187	0.0105	0.0088	0.0054	0.0036	0.0028	0.0015
5	−0.0123	0.0309	0.0222	0.0278	0.0133	0.0132	0.0069	0.0054	0.0037	0.0023
6	−0.0072	0.0440	0.0266	0.0387	0.0160	0.0185	0.0084	0.0076	0.0045	0.0033

is small or moderate (10 or even 50) the MSE of $\hat{C}_{pmk}$ is quite large compared with the corresponding actual value of C_{pmk}. For example, for $|\mu - T|/\sigma = 0$ and $d/\sigma = 6$, $C_{pmk} = 2.00$ while $\sqrt{\mathrm{MSE}(\hat{C}_{pmk})} = 0.559$ for $n = 10$ and $\sqrt{\mathrm{MSE}(\hat{C}_{pmk})} = 0.210$ for $n = 50$. This shows that it is important to construct confidence intervals or hypothesis tests.

3. Testing Process Capability

Using the index C_{pmk}, the engineers can access the process performance and monitor the manufacturing processes on a routine basis. To test whether a given process is capable, we can consider the following statistical testing hypotheses:

$$H_0\colon C_{pmk} \leq C \text{ (process is not capable)},$$

$$H_1\colon C_{pmk} > C \text{ (process is capable)}.$$

We define the test $\phi^*(x)$ as the following: $\phi^*(x) = 1$ if $\hat{C}_{pmk} > c_0$, and $\phi^*(x) = 0$ otherwise. Thus, the test ϕ^* rejects the null hypothesis $H_0(C_{pmk} \leq C)$ if $\hat{C}_{pmk} > c_0$, with type I error $\alpha(c_0) = \alpha$, the chance of incorrectly concluding an incapable process ($C_{pmk} \leq C$) as capable ($C_{pmk} > C$). Given values of α and C, the critical value c_0 can be obtained by solving the equation $P(\hat{C} \geq c_0 \,|\, C_{pmk} = C) = \alpha$ using available numerical methods. The derivation for solving the critical value c_0 is given in the Appendix A, from which we can find the critical value c_0 satisfying the following equation,

$$[2^{(n-1)/2}\Gamma((n-1)/2)]^{-1} \int_0^{\Delta} [\Phi(B(y)) - \Phi(A(y))] y^{(n-3)/2} e^{-y/2} dy = \alpha ,$$

where

$$Q = \frac{(\mu - T)}{\sigma}, \quad \Delta = \frac{n(3C\sqrt{1+Q^2} + |Q|)^2}{(9c_0^2)},$$

$$A(y) = \frac{1}{1-9c_0^2}\left\{\sqrt{n}(9c_0^2 Q - d_1) + \sqrt{n(9c_0^2 Q - d_1)^2 - (1-9c_0^2)[nd_1^2 - 9c_0^2(y + nQ^2)]}\right\},$$

$$B(y) = \frac{1}{1-9c_0^2}\left\{\sqrt{n}(9c_0^2 Q + d_2) - \sqrt{n(9c_0^2 Q + d_2)^2 - (1-9c_0^2)[nd_2^2 - 9c_0^2(y + nQ^2)]}\right\},$$

$$d_1 = 3C\sqrt{1+Q^2} + |Q| + Q, \quad \text{and} \quad d_2 = 3C\sqrt{1+Q^2} + |Q| - Q.$$

Using the techniques of the numerical integration, we can find the critical value c_0 for various values of n, C, Q, and α, which are summarized in Tables A1–A3 and Tables B1–B3. To find the exact value c_0 satisfying $P(\hat{C}_{pmk} \geq c_0 \,|\, C_{pmk} = C) = \alpha$ for given C and α, we can execute the *Turbo C* computer program (available from the authors). The program is accurate with an absolute error less than 10^{-4}.

The tables listed in Table 3 display critical values c_0 for $C = 1.00$ and 1.33 with sample sizes $n = 30(10)200$, $|Q| = 0.00(0.05)1.00$, and α-risk $= 0.01, 0.025, 0.05$. We note that the critical values c_0 for Q and $(-Q)$ are the same (for the proof see Appendix B). To test if the process meets the capability (quality) requirement,

Table 3. Guidelines for table selection of the critical values c_0.

C	Table	α	Sample Size n	$\lvert Q\rvert$
1.00	A1	0.01	30(10)200	0.00(0.05)1.00
1.00	A2	0.025	30(10)200	0.00(0.05)1.00
1.00	A3	0.05	30(10)200	0.00(0.05)1.00
1.33	B1	0.01	30(10)200	0.00(0.05)1.00
1.33	B2	0.025	30(10)200	0.00(0.05)1.00
1.33	B3	0.05	30(10)200	0.00(0.05)1.00

we first determine the value of C and the α-risk. Since the process parameters μ and σ are unknown, then Q is also unknown. But, similar with that for testing C_{pm} in Cheng[5] we can estimate Q by calculating the values $\hat{Q} = (\bar{X} - T)/S_n$ from the sample. Checking the appropriate table listed in Table 3, we may obtain the critical value c_0 based on given values of α-risk, C, $\hat{Q}$, and the sample size n. If the estimated value $\hat{C}_{pmk}$ is greater than the critical value c_0 ($\hat{C}_{pmk} > c_0$), then we conclude that the process meets the capability requirement ($C_{pmk} > C$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_{pmk} \leq C$.

The Procedure

Step 1. Decide the definition of "capable" (C, normally set to 1.00, or 1.33), and the α-risk (normally set to 0.01, 0.025, or 0.05), the chance of wrongly concluding an incapable process as capable.

Step 2. Calculate the values of $\hat{Q} = (\bar{X} - T)/S_n$ and $\hat{C}_{pmk}$ from the sample.

Step 3. Check the appropriate table listed in Table 3 and find the critical value c_0 based on α-risk, C, Q (estimated by $\hat{Q}$), and the sample size n.

Step 4. Conclude that the process is capable ($C_{pmk} > C$) if $\hat{C}_{pmk}$ value is greater than the critical value c_0 ($\hat{C}_{pmk} > c_0$). Otherwise, we do not have enough information to conclude that the process is capable.

In above testing procedure, we find the critical value c_0 based on Q (estimated by $\hat{Q}$). We can pick the greatest critical value among those corresponding to all Q-values in Tables A1–A3 and Tables B1–B3 for various values of n, C, and α. Then, we get the more conservative critical values, which are independent of Q, displayed in Tables 4(a) and 4(b). Now, we may obtain the critical value c_0 from Tables 4(a) and 4(b) based on given values of α-risk, C, and the sample size n. If the estimated value $\hat{C}_{pmk}$ is greater than the critical value c_0 ($\hat{C}_{pmk} > c_0$), then we conclude that the process meets the capability requirement ($C_{pmk} > C$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_{pmk} \leq C$.

Table 4(a). Critical values c_0 for $C = 1.00$, $n = 10(10)200$ and $\alpha = 0.01, 0.025, 0.05$.

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
10	2.161	1.902	1.712
20	1.678	1.544	1.440
30	1.513	1.416	1.340
40	1.425	1.347	1.285
50	1.369	1.303	1.249
60	1.330	1.271	1.224
70	1.301	1.248	1.205
80	1.278	1.229	1.189
90	1.259	1.214	1.177
100	1.244	1.202	1.167
110	1.231	1.191	1.158
120	1.219	1.182	1.151
130	1.210	1.174	1.144
140	1.201	1.167	1.138
150	1.193	1.161	1.133
160	1.186	1.155	1.129
170	1.180	1.150	1.124
180	1.174	1.145	1.121
190	1.169	1.141	1.117
200	1.165	1.137	1.114

Table 4(b). Critical values c_0 for $C = 1.33$, $n = 10(10)200$ and $a = 0.01, 0.025, 0.05$.

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
10	2.799	2.469	2.228
20	2.184	2.014	1.882
30	1.974	1.852	1.756
40	1.864	1.765	1.687
50	1.793	1.710	1.642
60	1.744	1.670	1.610
70	1.707	1.640	1.586
80	1.678	1.617	1.567
90	1.654	1.598	1.552
100	1.635	1.582	1.539
110	1.619	1.569	1.528

Table 4(b). (*Continued*).

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
120	1.604	1.557	1.518
130	1.592	1.547	1.510
140	1.581	1.539	1.503
150	1.572	1.531	1.497
160	1.563	1.524	1.491
170	1.555	1.517	1.485
180	1.548	1.511	1.481
190	1.542	1.506	1.476
200	1.536	1.501	1.472

4. Application of the Procedure

To demonstrate how we apply the proposed procedure to the actual data and test whether the process is capable, we collect some sample data from Nuera, a manufacturer and supplier in Taiwan exporting high-end audio speaker drivers. One characteristic, which critically determines the bass performance, musical image, clarity and cleanness of the sound, transparence, and compliance (excursion movement) of the mid-range, full-range, or subwoofer driver units, is F_o (the free-air resonance). For each model of drivers, a unique production specification (LSL, T, USL) for F_o is set. One particular model of the 3-inch full-range drivers requires the F_o value 80 Hz with ± 10 Hz tolerance, i.e., the production specifications for the 3-inch full-range drivers are the following: LSL $= 70$, USL $= 90$, and $T = 80$.

Currently, the Nuera's Quality Control Center is implementing a quality program which requires all models of audio-speaker driver products satisfying the quality condition "capable", or $1.00 \leq C_{pmk} < 1.33$ in the current stage. The collected sample data (a total of 100 observations) are displayed in Table 5.

Table 5. Sample data of F_o (100 observations).

77	76	77	83	74	70	80	78	85	79
74	73	77	73	75	82	79	80	82	80
79	72	82	80	73	74	75	81	82	79
82	75	78	81	77	76	77	76	78	75
82	76	77	84	78	80	78	75	78	82
83	72	76	78	78	79	80	81	79	77
71	76	84	76	77	81	78	74	78	75
77	77	81	80	85	80	75	73	75	76
81	78	85	77	73	74	84	80	73	77
75	77	76	80	76	75	79	80	80	77

To test whether the process is "capable", we first calculate $d = (\text{USL}-\text{LSL})/2 = 10$, sample mean $\bar{X} = \Sigma_{i=1}^{1} X_i/n = 77.88$, and sample standard deviation $S_n = \{\Sigma_{i=1}^{n}(X_i - \bar{X})^2/n\}^{1/2} = 3.24$. Then, we calculate $\hat{Q} = (\bar{X} - T)/S_n = -0.65$ and $\hat{C}_{pmk} = 0.68$. Checking Table A1 with $C = 1.00$, $\alpha = 0.01$, and sample size $n = 100$, we find the critical value $c_0 = 1.242$ for $|Q| = 0.65$. Clearly, we cannot conclude that the process is capable since $\hat{C}_{pmk}$ value is less than the critical value c_0. We note that we have the same conclusion based on the more conservative critical value $c_0 = 1.244$ from Table 4(a) for $C = 1.00$, $\alpha = 0.01$, and $n = 100$.

The quality condition of such a process was considered to be unsatisfactory in the company. Some quality improvement activities involving Taguchi's parameter design were initiated to identify the significant factors causing the process failing to meet the company's requirement set for the current stage, $1.00 \le C_{pmk} < 1.33$. Consequently, some machine settings for the assembly of the drivers including the gluing of the spiders and Pulux edges as well as other component sticking operations were adjusted. To check whether the quality of the adjusted process is satisfactory, a new sample of 100 observations from the adjusted process was collected, yielding the measurements displayed in Table 6. Specifications, process capability requirements, and the α-risk remained the same.

Table 6. Sample data of F_o (100 observations).

81	80	82	79	78	76	78	78	76	81
83	78	81	85	81	78	79	79	80	82
79	79	82	78	82	80	75	85	80	80
80	75	81	78	82	84	76	78	80	79
82	82	78	78	82	78	82	80	82	83
81	78	83	81	82	79	80	79	81	82
79	80	82	77	81	80	81	81	75	76
83	86	82	79	82	85	80	80	77	75
78	85	81	79	81	83	78	78	80	80
79	76	77	74	85	83	76	80	75	82

We perform the same calculations over the new sample of 100 observations. We first calculate $\bar{X} = 79.92$, $S_n = 2.58$, $\hat{Q} = -0.03$, and $\hat{C}_{pmk} = 1.28$. To obtain the corresponding value of c_0, we check Table A1 and obtain the critical value $c_0 = 1.173$ for $|Q| = 0.00$, and $c_0 = 1.191$ for $|Q| = 0.05$. Using interpolation technique, we obtain the corresponding critical value $c_0 = 1.184$ for $|Q| = 0.03$. Since $\hat{C}_{pmk}$ value is greater than the critical value c_0, we conclude that the adjusted process meets the company's requirement. We note that we have the same conclusion based on the more conservative critical value $c_0 = 1.244$ from Table 4(a) for $C = 1.00$, $\alpha = 0.01$, and $n = 100$.

5. Conclusions

In this paper, we develop a simple procedure using a natural estimator of C_{pmk} proposed by Pearn et al.[3] based on the theory of testing hypothesis. The procedure provides the practitioners an easy way to test the process capability. Factory engineers or supervisors can use the procedure to determine whether their process meets the capability requirements preset in the factory. The proposed procedure is applied to an audio-speaker driver manufacturing process which we studied. Problems causing the process failing to meet the company's requirement were troubleshooted. Quality improvement activities were conducted and machine settings were adjusted. As a result, problems causing the process failing to meet the company's requirement were successfully resolved, and the capability of the adjusted process is concluded to be satisfactory.

Appendix A

Consider the following statistical testing hypothesis for C_{pmk}:

$$H_0 : C_{pmk} \le C\,,$$

$$H_1 : C_{pmk} > C\,.$$

Given a value of C, the p-value corresponding to c_0 is

$$\begin{aligned}
&P\{\hat{C}_{pmk} \ge c_0 \,|\, C_{pmk} = C\} \\
&\quad = P\left\{ \frac{\bar{X} - \mathrm{LSL}}{3\sqrt{S_n^2 + (\bar{X} - T)^2}} \ge c_0, \frac{\mathrm{USL} - \bar{X}}{3\sqrt{S_n^2 + (\bar{X} - T)^2}} \ge c_0 | C_{pmk} = C \right\} \\
&\quad = P\left\{ \frac{\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} + \dfrac{\mu - \mathrm{LSL}}{\sigma/\sqrt{n}}}{3\sqrt{\dfrac{nS_n^2}{\sigma^2} + \left(\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} + \dfrac{\mu - T}{\sigma/\sqrt{n}}\right)^2}} \right. \\
&\qquad \left. \ge c_0, \frac{\dfrac{\mu - \bar{X}}{\sigma/\sqrt{n}} + \dfrac{\mathrm{USL} - \mu}{\sigma/\sqrt{n}}}{3\sqrt{\frac{nS_n^2}{\sigma^2} + \left(\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} + \dfrac{\mu - T}{\sigma/\sqrt{n}}\right)^2}} \ge c_0 | C_{pmk} = C \right\} \\
&\quad = P\left\{ \frac{Z + \sqrt{n}d_1}{3\sqrt{Y + (Z + \sqrt{n}Q)^2}} \ge c_0, \frac{Z + \sqrt{n}d_2}{3\sqrt{Y + (Z + \sqrt{n}Q)^2}} \ge c_0 | C_{pmk} = C \right\},
\end{aligned}$$

where $Y = nS_n^2/\sigma^2$, $Z = n^{1/2}(\bar{X} - \mu)/\sigma$, $d_1 = (\mu - \mathrm{LSL})/\sigma$, $d_2 = (\mathrm{USL} - \mu)/\sigma$ and $Q = (\mu - T)/\sigma$. If $T = m$, then $\mathrm{USL} - T = T - \mathrm{LSL} = d$, and

$$C_{pmk} = \frac{d - |\mu - T|}{3\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d/\sigma - |Q|}{3\sqrt{1 + Q^2}}\,.$$

For $C_{pmk} = C$, d/σ can be solved as $d/\sigma = 3C\sqrt{1+Q^2} + |Q|$. Then, d_1 and d_2 can be expressed as functions of C and Q:

$$d_1 = \frac{\mu - \text{LSL}}{\sigma} = \frac{T - \text{LSL}}{\sigma} + \frac{\mu - T}{\sigma} = \frac{d}{\sigma} + Q = 3C\sqrt{1+Q^2} + |Q| + Q\,,$$

$$d_2 = \frac{\text{USL} - \mu}{\sigma} = \frac{\text{USL} - T}{\sigma} - \frac{\mu - T}{\sigma} = \frac{d}{\sigma} - Q = 3C\sqrt{1+Q^2} + |Q| - Q\,.$$

Therefore, we have:

$$\begin{aligned}
&P\{\hat{C}_{pmk} \geq c_0 \,|\, C_{pmk} = C\} \\
&\quad = P\left\{ \frac{(Z+\sqrt{n}d_1)^2}{Y + (Z+\sqrt{n}Q)^2} \geq 9c_0^2, \frac{(Z+\sqrt{n}d_1)^2}{Y + (Z+\sqrt{n}Q)^2} \geq 9c_0 | C_{pmk} = C \right\}, \\
&\quad = P\{9c_0^2 Y \leq (1-9c_0^2)Z^2 + 2\sqrt{n}(d_1 - 9c_0^2 Q)Z + nd_1^2 - 9nc_0^2Q^2\,, \\
&\qquad 9c_0^2 Y \leq (1-9c_0^2)Z^2 - 2\sqrt{n}(d_2 - 9c_0^2 Q)Z + nd_2^2 - 9nc_0^2Q^2 \,|\, C_{pmk} = C\}\,.
\end{aligned}$$

In most applications, the value of C would be greater than 1.00. If $c_0 \leq 1/3$ for $C \geq 1.00$, then the corresponding p-value is far greater than the value of α-risk, we would accept "H_0: process is not capable" for such small value of c_0. For $c_0 > 1/3$, we have:

$$P\{\hat{C}_{pmk} \geq c_0 \,|\, C_{pmk} = C\} = P\{0 < Y \leq \Delta,\ A(Y) \leq Z \leq B(Y) \,|\, C_{pmk} = C\}\,,$$

where

$$\Delta = n\left(3C\sqrt{1+Q^2} + |Q|\right)^2 / (9c_0^2)\,,$$

$$\begin{aligned}
A(Y) = \frac{1}{1-9c_0^2}\Big\{ &\sqrt{n}(9c_0^2 Q - d_1) \\
&+ \sqrt{n(9c_0^2Q - d_1)^2 - (1-9c_0^2)[nd_1^2 - 9c_0^2(Y + nQ^2)]} \Big\}\,,
\end{aligned}$$

$$\begin{aligned}
B(Y) = \frac{1}{1-9c_0^2}\Big\{ &\sqrt{n}(9c_0^2 Q + d_2) \\
&- \sqrt{n(9c_0^2Q + d_2)^2 - (1-9c_0^2)[nd_2^2 - 9c_0^2(Y + nQ^2)]} \Big\}\,.
\end{aligned}$$

Under the assumption of normality, Y and Z are independent random variables. In fact, Y is distributed as a central chi-square distribution with $n-1$ degrees of freedom and Z is distributed as the standard normal distribution. Let $f_Y(y)$ and $\phi(z)$ be the probability density functions of Y and Z, respectively. Then, the joint probability density function of Y and Z is $f_{Y,Z}(y,z) = f_Y(y)\phi(z)$. Hence, the

p-value becomes

$$
\begin{aligned}
&P\{\hat{C}_{pmk} \geq c_0 \,|\, C_{pmk} = C\} \\
&\quad = P\{0 < Y \leq \Delta, A(Y) \leq Z \leq B(Y) \,|\, C_{pmk} = C\} \\
&\quad = \int_0^{\Delta} \int_{A(y)}^{B(y)} f_{Y,Z}(y,z) dz dy \\
&\quad = \int_0^{\Delta} \int_{A(y)}^{B(y)} f_Y(y)\phi(z) dz dy \\
&\quad = \int_0^{\Delta} f_Y(y) \int_{A(y)}^{B(y)} \phi(z) dz dy \\
&\quad = \int_0^{\Delta} f_Y(y)(\phi(B(y)) - \phi(A(y))) dy \\
&\quad = \frac{1}{\Gamma[(n-1)/2]2^{(n-1)/2}} \int_0^{\Delta} (\Phi(B(y)) - \Phi(A(y))) y^{(n-3)/2} e^{-y/2} dy \,,
\end{aligned}
$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution $N(0,1)$. Now, given values of C, Q, n, and α, the critical value c_0 can be obtained by solving the equation

$$P\{\hat{C}_{pmk} \geq c_0 \,|\, C_{pmk} = C\} = \alpha \,,$$

or

$$\frac{1}{\Gamma[(n-1)/2]2^{(n-1)/2}} \int_0^{\Delta} (\Phi(B(y)) - \Phi(A(y))) y^{(n-3)/2} e^{-y/2} dy = \alpha \,.$$

Appendix B

We will show that given the same values of C, n, and α the critical value c_0 for Q and $(-Q)$ are the same. Since $d_1 = (d/\sigma + Q)$ and $d_2 = (d/\sigma - Q)$ are functions of Q, then setting $d_1(Q) = (d/\sigma + Q)$ and $d_2(Q) = (d/\sigma - Q)$, we have $d_1(-Q) = d_2(Q) = d_2$ and $d_2(-Q) = d_1(Q) = d_1$. In addition, $\Delta = n(3C\sqrt{1+Q^2} + |Q|)^2/(9c_0^2)$ is also a function of Q. Setting $\Delta(Q) = n(3C\sqrt{1+Q^2} + |Q|)^2/(9c_0^2)$, we have $\Delta(-Q) = \Delta(Q) = \Delta$. Furthermore, both $A(Y)$ and $B(Y)$ are functions of Q. If we set functions $A(Q,Y) = A(Y)$ and $B(Q,Y) = B(Y)$ respectively, then we have:

$$
\begin{aligned}
A(-Q,Y) = -\frac{1}{1-9c_0^2} \Big\{ &\sqrt{n}(9c_0^2 Q + d_2) \\
&- \sqrt{n(9c_0^2 Q + d_2)^2 - (1-9c_0^2)[nd_2^2 - 9c_0^2(Y + nQ^2)]} \Big\} \\
= -B(Y) \,,&
\end{aligned}
$$

Table A1. Critical values c_0 for $C = 1.00$ with $|Q| = 0.00(0.05)1.00$, and sample sizes $n = 30(10)200$ for α-risk $= 0.01$.

$\|Q\|$	$n = 30$	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.00	1.375	1.308	1.266	1.237	1.215	1.198	1.184	1.173	1.163	1.155	1.148	1.142	1.136	1.131	1.127	1.123	1.119	1.116
0.05	1.397	1.329	1.286	1.256	1.234	1.217	1.203	1.191	1.181	1.173	1.166	1.159	1.153	1.148	1.144	1.139	1.136	1.132
0.10	1.419	1.349	1.305	1.274	1.251	1.233	1.218	1.206	1.196	1.187	1.179	1.173	1.166	1.161	1.156	1.152	1.147	1.144
0.15	1.438	1.366	1.321	1.289	1.265	1.246	1.231	1.218	1.207	1.198	1.189	1.182	1.176	1.170	1.164	1.159	1.155	1.151
0.20	1.456	1.382	1.334	1.301	1.276	1.256	1.240	1.226	1.215	1.205	1.196	1.188	1.181	1.175	1.170	1.164	1.160	1.155
0.25	1.471	1.395	1.346	1.311	1.284	1.264	1.247	1.232	1.220	1.210	1.201	1.193	1.185	1.179	1.173	1.168	1.163	1.158
0.30	1.485	1.405	1.354	1.318	1.291	1.269	1.251	1.237	1.224	1.213	1.204	1.196	1.188	1.182	1.175	1.170	1.165	1.160
0.35	1.495	1.413	1.361	1.323	1.295	1.273	1.255	1.240	1.227	1.216	1.206	1.198	1.190	1.184	1.178	1.172	1.167	1.162
0.40	1.503	1.419	1.365	1.327	1.298	1.275	1.257	1.242	1.229	1.218	1.208	1.200	1.192	1.185	1.179	1.173	1.168	1.163
0.45	1.509	1.423	1.368	1.329	1.300	1.277	1.258	1.243	1.230	1.219	1.209	1.201	1.193	1.186	1.180	1.174	1.169	1.164
0.50	1.512	1.425	1.369	1.330	1.301	1.278	1.259	1.244	1.231	1.219	1.210	1.201	1.193	1.186	1.180	1.174	1.169	1.165
0.55	1.513	1.425	1.369	1.330	1.301	1.278	1.259	1.244	1.231	1.219	1.210	1.201	1.193	1.186	1.180	1.174	1.169	1.165
0.60	1.512	1.424	1.368	1.329	1.300	1.277	1.258	1.243	1.230	1.219	1.209	1.200	1.193	1.186	1.180	1.174	1.169	1.164
0.65	1.510	1.422	1.367	1.328	1.298	1.275	1.257	1.242	1.229	1.217	1.208	1.199	1.192	1.185	1.179	1.173	1.168	1.163
0.70	1.506	1.419	1.364	1.325	1.296	1.273	1.255	1.240	1.227	1.216	1.206	1.198	1.190	1.183	1.177	1.171	1.166	1.162
0.75	1.502	1.415	1.360	1.322	1.293	1.271	1.252	1.237	1.225	1.214	1.204	1.196	1.188	1.182	1.176	1.170	1.165	1.161
0.80	1.496	1.410	1.356	1.318	1.290	1.268	1.250	1.235	1.222	1.211	1.202	1.194	1.186	1.179	1.173	1.168	1.163	1.158
0.85	1.489	1.405	1.351	1.314	1.286	1.264	1.246	1.232	1.219	1.208	1.199	1.191	1.183	1.177	1.171	1.165	1.160	1.156
0.90	1.481	1.398	1.346	1.309	1.282	1.260	1.243	1.228	1.216	1.205	1.196	1.188	1.181	1.175	1.169	1.164	1.159	1.155
0.95	1.473	1.392	1.340	1.304	1.277	1.256	1.239	1.225	1.213	1.202	1.194	1.186	1.179	1.172	1.167	1.162	1.157	1.153
1.00	1.464	1.385	1.334	1.299	1.272	1.252	1.235	1.221	1.209	1.199	1.190	1.182	1.175	1.168	1.162	1.157	1.152	1.147

Table A2. Critical values c_0 for $C = 1.00$ with $|Q| = 0.00(0.05)1.00$, and sample sizes $n = 30(10)200$ for α-risk = 0.025.

$\|Q\|$	$n = 30$	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.00	1.294	1.243	1.211	1.188	1.171	1.158	1.147	1.139	1.131	1.125	1.119	1.114	1.110	1.106	1.102	1.099	1.096	1.093
0.05	1.316	1.264	1.231	1.207	1.190	1.176	1.165	1.156	1.148	1.142	1.136	1.131	1.126	1.122	1.118	1.115	1.112	1.109
0.10	1.336	1.282	1.248	1.224	1.206	1.191	1.180	1.170	1.162	1.155	1.149	1.143	1.138	1.134	1.130	1.126	1.123	1.120
0.15	1.354	1.298	1.262	1.237	1.218	1.203	1.191	1.181	1.172	1.164	1.158	1.152	1.146	1.141	1.137	1.133	1.129	1.126
0.20	1.369	1.312	1.275	1.248	1.228	1.212	1.199	1.188	1.179	1.170	1.163	1.157	1.151	1.146	1.141	1.137	1.133	1.130
0.25	1.383	1.323	1.284	1.256	1.235	1.218	1.205	1.193	1.183	1.174	1.167	1.160	1.154	1.149	1.144	1.140	1.136	1.132
0.30	1.394	1.332	1.291	1.262	1.240	1.223	1.208	1.196	1.186	1.177	1.170	1.163	1.157	1.151	1.146	1.142	1.138	1.134
0.35	1.403	1.339	1.297	1.266	1.244	1.226	1.211	1.199	1.188	1.179	1.171	1.164	1.158	1.153	1.148	1.143	1.139	1.135
0.40	1.409	1.343	1.300	1.269	1.246	1.228	1.213	1.200	1.190	1.181	1.173	1.166	1.160	1.154	1.149	1.144	1.140	1.136
0.45	1.414	1.346	1.302	1.271	1.247	1.229	1.214	1.201	1.191	1.182	1.174	1.167	1.160	1.155	1.150	1.145	1.141	1.137
0.50	1.416	1.347	1.303	1.271	1.248	1.229	1.214	1.202	1.191	1.182	1.174	1.167	1.161	1.155	1.150	1.145	1.141	1.137
0.55	1.416	1.347	1.303	1.271	1.248	1.229	1.214	1.202	1.191	1.182	1.174	1.167	1.160	1.155	1.150	1.145	1.141	1.137
0.60	1.416	1.346	1.302	1.271	1.247	1.228	1.213	1.201	1.190	1.181	1.173	1.166	1.160	1.154	1.149	1.145	1.140	1.136
0.65	1.413	1.344	1.300	1.269	1.246	1.227	1.212	1.200	1.189	1.180	1.172	1.165	1.159	1.153	1.148	1.144	1.140	1.136
0.70	1.410	1.342	1.298	1.267	1.244	1.225	1.211	1.198	1.188	1.179	1.171	1.164	1.158	1.152	1.147	1.143	1.138	1.135
0.75	1.406	1.338	1.295	1.264	1.241	1.223	1.208	1.196	1.186	1.177	1.169	1.162	1.156	1.151	1.146	1.141	1.137	1.133
0.80	1.401	1.334	1.291	1.261	1.238	1.221	1.206	1.194	1.184	1.175	1.167	1.160	1.154	1.149	1.144	1.140	1.136	1.132
0.85	1.396	1.330	1.287	1.258	1.235	1.218	1.203	1.191	1.181	1.173	1.165	1.158	1.152	1.147	1.142	1.138	1.134	1.130
0.90	1.389	1.325	1.283	1.254	1.232	1.214	1.200	1.189	1.179	1.170	1.163	1.156	1.150	1.145	1.140	1.136	1.132	1.128
0.95	1.383	1.319	1.279	1.250	1.228	1.211	1.197	1.186	1.176	1.167	1.160	1.154	1.148	1.143	1.138	1.134	1.130	1.126
1.00	1.376	1.314	1.274	1.245	1.224	1.207	1.194	1.183	1.173	1.164	1.157	1.150	1.144	1.139	1.134	1.130	1.125	1.122

Table A3. Critical values c_0 for $C = 1.00$ with $|Q| = 0.00(0.05)1.00$, and sample sizes $n = 30(10)200$ for α-risk $= 0.05$.

$\|Q\|$	$n = 30$	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.00	1.231	1.192	1.167	1.149	1.136	1.126	1.117	1.110	1.104	1.099	1.095	1.091	1.088	1.084	1.082	1.079	1.077	1.075
0.05	1.251	1.211	1.186	1.167	1.154	1.143	1.134	1.127	1.121	1.116	1.111	1.107	1.103	1.100	1.097	1.095	1.092	1.090
0.10	1.270	1.228	1.202	1.183	1.168	1.157	1.148	1.140	1.134	1.128	1.123	1.119	1.115	1.111	1.108	1.105	1.102	1.100
0.15	1.286	1.243	1.215	1.195	1.180	1.168	1.158	1.150	1.143	1.136	1.131	1.126	1.122	1.118	1.114	1.111	1.108	1.105
0.20	1.301	1.255	1.226	1.205	1.189	1.176	1.165	1.156	1.148	1.142	1.136	1.131	1.126	1.122	1.118	1.114	1.111	1.108
0.25	1.313	1.265	1.234	1.212	1.195	1.181	1.170	1.160	1.152	1.145	1.139	1.133	1.128	1.124	1.120	1.116	1.113	1.110
0.30	1.322	1.273	1.240	1.217	1.199	1.184	1.173	1.163	1.154	1.147	1.141	1.135	1.130	1.126	1.122	1.118	1.114	1.111
0.35	1.329	1.278	1.244	1.220	1.201	1.187	1.175	1.165	1.156	1.149	1.142	1.137	1.132	1.127	1.123	1.119	1.116	1.113
0.40	1.335	1.282	1.247	1.222	1.203	1.188	1.176	1.166	1.157	1.150	1.143	1.138	1.132	1.128	1.124	1.120	1.116	1.113
0.45	1.338	1.284	1.248	1.223	1.204	1.189	1.177	1.167	1.158	1.151	1.144	1.138	1.133	1.128	1.124	1.121	1.117	1.114
0.50	1.339	1.285	1.249	1.224	1.205	1.189	1.177	1.167	1.158	1.151	1.144	1.138	1.133	1.129	1.124	1.121	1.117	1.114
0.55	1.340	1.284	1.249	1.223	1.204	1.189	1.177	1.167	1.158	1.151	1.144	1.138	1.133	1.129	1.124	1.121	1.117	1.114
0.60	1.339	1.283	1.248	1.223	1.204	1.189	1.176	1.166	1.158	1.150	1.144	1.138	1.133	1.128	1.124	1.120	1.117	1.114
0.65	1.337	1.282	1.246	1.221	1.202	1.187	1.175	1.165	1.157	1.149	1.143	1.137	1.132	1.127	1.123	1.119	1.116	1.113
0.70	1.334	1.279	1.244	1.220	1.201	1.186	1.174	1.164	1.155	1.148	1.142	1.136	1.131	1.126	1.122	1.119	1.115	1.112
0.75	1.330	1.277	1.242	1.217	1.199	1.184	1.172	1.162	1.154	1.147	1.140	1.135	1.130	1.125	1.121	1.117	1.114	1.111
0.80	1.326	1.273	1.239	1.215	1.196	1.182	1.170	1.160	1.152	1.145	1.138	1.133	1.128	1.123	1.119	1.115	1.112	1.108
0.85	1.322	1.269	1.236	1.212	1.194	1.179	1.168	1.158	1.150	1.143	1.137	1.131	1.127	1.122	1.118	1.115	1.112	1.108
0.90	1.317	1.265	1.232	1.209	1.191	1.177	1.165	1.156	1.148	1.141	1.135	1.130	1.125	1.121	1.117	1.114	1.111	1.107
0.95	1.311	1.261	1.228	1.205	1.188	1.174	1.163	1.153	1.145	1.138	1.132	1.127	1.122	1.117	1.113	1.110	1.106	1.105
1.00	1.306	1.256	1.224	1.202	1.185	1.171	1.160	1.151	1.143	1.136	1.130	1.125	1.120	1.115	1.111	1.108	1.105	1.103

Table B1. Critical values c_0 for $C = 1.33$ with $|Q| = 0.00(0.05)1.00$, and sample sizes $n = 30(10)200$ for α-risk $= 0.01$.

$\lvert Q \rvert$	$n = 30$	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.00	1.840	1.750	1.693	1.653	1.623	1.600	1.582	1.566	1.553	1.542	1.532	1.524	1.516	1.150	1.503	1.498	1.493	1.488
0.05	1.863	1.771	1.713	1.673	1.642	1.619	1.600	1.584	1.571	1.560	1.550	1.541	1.533	1.562	1.520	1.514	1.509	1.504
0.10	1.884	1.790	1.731	1.690	1.659	1.635	1.615	1.599	1.585	1.573	1.563	1.554	1.546	1.539	1.532	1.526	1.520	1.515
0.15	1.904	1.808	1.747	1.704	1.673	1.647	1.627	1.610	1.596	1.583	1.573	1.563	1.554	1.547	1.540	1.533	1.528	1.522
0.20	1.921	1.823	1.760	1.716	1.683	1.657	1.636	1.618	1.603	1.590	1.579	1.569	1.560	1.552	1.545	1.538	1.532	1.526
0.25	1.936	1.836	1.771	1.726	1.692	1.665	1.643	1.624	1.609	1.595	1.584	1.573	1.564	1.556	1.548	1.542	1.535	1.530
0.30	1.949	1.846	1.780	1.733	1.698	1.670	1.647	1.628	1.612	1.599	1.587	1.576	1.567	1.558	1.551	1.544	1.537	1.532
0.35	1.959	1.853	1.786	1.738	1.702	1.674	1.651	1.631	1.615	1.601	1.589	1.579	1.569	1.561	1.553	1.546	1.539	1.534
0.40	1.967	1.859	1.790	1.741	1.705	1.676	1.653	1.634	1.617	1.603	1.591	1.580	1.571	1.562	1.554	1.547	1.541	1.535
0.45	1.972	1.862	1.792	1.743	1.706	1.677	1.654	1.635	1.618	1.604	1.592	1.581	1.572	1.563	1.555	1.548	1.542	1.536
0.50	1.974	1.864	1.793	1.744	1.707	1.678	1.654	1.635	1.619	1.604	1.592	1.581	1.572	1.563	1.555	1.548	1.542	1.536
0.55	1.974	1.863	1.793	1.743	1.706	1.677	1.654	1.635	1.618	1.604	1.592	1.581	1.571	1.563	1.555	1.548	1.541	1.535
0.60	1.973	1.862	1.791	1.742	1.705	1.676	1.653	1.634	1.617	1.603	1.591	1.580	1.570	1.562	1.554	1.547	1.541	1.534
0.65	1.969	1.858	1.788	1.739	1.703	1.674	1.651	1.632	1.615	1.601	1.589	1.579	1.569	1.560	1.553	1.546	1.539	1.534
0.70	1.964	1.854	1.785	1.736	1.700	1.671	1.648	1.629	1.613	1.599	1.587	1.577	1.567	1.559	1.551	1.544	1.538	1.532
0.75	1.957	1.849	1.780	1.732	1.696	1.668	1.645	1.626	1.610	1.597	1.584	1.574	1.565	1.556	1.549	1.542	1.535	1.529
0.80	1.949	1.842	1.775	1.727	1.692	1.664	1.641	1.623	1.607	1.593	1.581	1.571	1.562	1.554	1.546	1.540	1.534	1.528
0.85	1.941	1.835	1.768	1.722	1.687	1.659	1.637	1.619	1.603	1.590	1.578	1.568	1.559	1.551	1.543	1.537	1.530	1.525
0.90	1.931	1.827	1.762	1.716	1.681	1.654	1.633	1.615	1.599	1.586	1.575	1.564	1.555	1.547	1.540	4.533	1.527	1.521
0.95	1.921	1.819	1.755	1.709	1.676	1.649	1.628	1.610	1.595	1.582	1.571	1.561	1.552	1.544	1.537	1.531	1.527	1.520
1.00	1.910	1.810	1.747	1.703	1.670	1.644	1.623	1.605	1.591	1.578	1.567	1.558	1.549	1.541	1.535	1.528	1.523	1.518

Table B2. Critical values c_0 for $C = 1.33$ with $|Q| = 0.00(0.05)1.00$, and sample sizes $n = 30(10)200$ for α-risk $= 0.025$.

$\lvert Q\rvert$	$n = 30$	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.00	1.734	1.664	1.620	1.589	1.566	1.548	1.533	1.521	1.510	1.502	1.494	1.487	1.481	1.476	1.471	1.466	1.462	1.458
0.05	1.756	1.685	1.640	1.608	1.584	1.566	1.551	1.538	1.528	1.519	1.511	1.504	1.497	1.492	1.487	1.482	1.478	1.474
0.10	1.775	1.703	1.657	1.624	1.600	1.580	1.565	1.552	1.541	1.531	1.523	1.516	1.509	1.503	1.498	1.493	1.488	1.484
0.15	1.793	1.719	1.671	1.637	1.612	1.592	1.576	1.562	1.550	1.540	1.531	1.524	1.517	1.510	1.505	1.499	1.495	1.490
0.20	1.808	1.732	1.683	1.648	1.621	1.600	1.583	1.569	1.557	1.546	1.537	1.529	1.521	1.515	1.509	1.503	1.498	1.494
0.25	1.822	1.743	1.692	1.656	1.628	1.607	1.589	1.574	1.561	1.550	1.541	1.532	1.525	1.518	1.512	1.506	1.501	1.496
0.30	1.833	1.752	1.699	1.662	1.633	1.611	1.592	1.577	1.564	1.553	1.543	1.535	1.527	1.520	1.514	1.508	1.503	1.498
0.35	1.841	1.758	1.704	1.666	1.637	1.614	1.595	1.580	1.566	1.555	1.545	1.536	1.529	1.522	1.515	1.510	1.504	1.500
0.40	1.847	1.762	1.707	1.668	1.639	1.616	1.597	1.581	1.568	1.556	1.546	1.538	1.530	1.523	1.516	1.511	1.505	1.501
0.45	1.851	1.765	1.709	1.670	1.640	1.617	1.598	1.582	1.569	1.557	1.547	1.538	1.531	1.523	1.517	1.511	1.506	1.501
0.50	1.852	1.765	1.710	1.670	1.640	1.617	1.598	1.582	1.569	1.557	1.547	1.539	1.531	1.524	1.517	1.511	1.506	1.501
0.55	1.852	1.765	1.709	1.670	1.640	1.617	1.598	1.582	1.569	1.557	1.547	1.538	1.530	1.523	1.517	1.511	1.506	1.500
0.60	1.851	1.763	1.708	1.668	1.639	1.615	1.597	1.581	1.568	1.556	1.546	1.537	1.530	1.523	1.516	1.510	1.505	1.500
0.65	1.847	1.761	1.705	1.666	1.637	1.614	1.595	1.579	1.566	1.555	1.545	1.536	1.528	1.521	1.515	1.509	1.504	1.499
0.70	1.843	1.757	1.702	1.663	1.634	1.611	1.593	1.577	1.564	1.553	1.543	1.534	1.527	1.520	1.513	1.508	1.503	1.498
0.75	1.838	1.753	1.698	1.660	1.631	1.608	1.590	1.575	1.562	1.551	1.541	1.532	1.525	1.518	1.512	1.506	1.501	1.496
0.80	1.831	1.747	1.694	1.656	1.627	1.605	1.587	1.572	1.559	1.548	1.538	1.530	1.522	1.516	1.509	1.504	1.499	1.494
0.85	1.824	1.741	1.689	1.651	1.623	1.601	1.583	1.569	1.556	1.545	1.536	1.527	1.520	1.513	1.507	1.502	1.497	1.492
0.90	1.816	1.735	1.683	1.647	1.619	1.597	1.580	1.565	1.553	1.542	1.533	1.524	1.517	1.511	1.505	1.499	1.494	1.490
0.95	1.808	1.728	1.677	1.641	1.614	1.593	1.576	1.561	1.549	1.539	1.530	1.521	1.514	1.508	1.502	1.497	1.492	1.487
1.00	1.799	1.721	1.671	1.636	1.609	1.589	1.572	1.557	1.546	1.535	1.526	1.518	1.511	1.505	1.499	1.494	1.489	1.485

Table B3. Table B3. Critical values c_0 for $C = 1.33$ with $|Q| = 0.00(0.05)1.00$, and sample sizes $n = 30(10)200$ for α-risk $= 0.05$.

$\lvert Q\rvert$	$n = 30$	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
0.00	1.650	1.596	1.561	1.537	1.519	1.505	1.493	1.483	1.475	1.468	1.462	1.457	1.452	1.447	1.443	1.440	1.437	1.434
0.05	1.670	1.615	1.580	1.555	1.537	1.522	1.510	1.500	1.492	1.484	1.478	1.472	1.467	1.463	1.459	1.455	1.452	1.449
0.10	1.689	1.632	1.596	1.570	1.551	1.536	1.523	1.513	1.504	1.496	1.490	1.484	1.478	1.474	1.469	1.465	1.462	1.458
0.15	1.705	1.647	1.609	1.583	1.562	1.546	1.533	1.522	1.513	1.504	1.497	1.491	1.485	1.480	1.475	1.471	1.467	1.463
0.20	1.719	1.659	1.620	1.592	1.571	1.554	1.540	1.528	1.518	1.509	1.502	1.495	1.489	1.484	1.479	1.474	1.470	1.466
0.25	1.731	1.668	1.628	1.599	1.576	1.559	1.544	1.532	1.522	1.513	1.505	1.498	1.492	1.486	1.481	1.476	1.472	1.468
0.30	1.740	1.676	1.634	1.603	1.580	1.562	1.547	1.535	1.524	1.515	1.507	1.500	1.494	1.488	1.483	1.478	1.474	1.470
0.35	1.747	1.681	1.638	1.607	1.583	1.564	1.549	1.537	1.526	1.517	1.508	1.501	1.495	1.489	1.484	1.479	1.475	1.471
0.40	1.752	1.684	1.640	1.609	1.585	1.566	1.551	1.538	1.527	1.518	1.510	1.502	1.496	1.490	1.485	1.480	1.476	1.472
0.45	1.755	1.686	1.641	1.610	1.586	1.567	1.551	1.539	1.528	1.518	1.510	1.503	1.497	1.491	1.486	1.481	1.476	1.472
0.50	1.756	1.687	1.642	1.610	1.586	1.567	1.552	1.539	1.528	1.518	1.510	1.503	1.497	1.491	1.485	1.481	1.476	1.472
0.55	1.755	1.686	1.641	1.609	1.585	1.567	1.551	1.538	1.528	1.518	1.510	1.503	1.496	1.490	1.485	1.480	1.476	1.472
0.60	1.754	1.684	1.640	1.608	1.584	1.565	1.550	1.537	1.527	1.517	1.509	1.502	1.496	1.490	1.485	1.480	1.475	1.472
0.65	1.751	1.682	1.638	1.606	1.583	1.564	1.549	1.536	1.525	1.516	1.508	1.501	1.495	1.489	1.484	1.480	1.475	1.471
0.70	1.747	1.679	1.635	1.604	1.580	1.562	1.547	1.534	1.524	1.515	1.507	1.499	1.493	1.487	1.482	1.479	1.473	1.469
0.75	1.743	1.675	1.632	1.601	1.578	1.560	1.545	1.532	1.522	1.513	1.505	1.498	1.492	1.486	1.481	1.477	1.472	1.468
0.80	1.737	1.671	1.628	1.598	1.575	1.557	1.542	1.530	1.519	1.510	1.53	1.496	1.490	1.484	1.479	1.476	1.471	1.467
0.85	1.732	1.666	1.624	1.594	1.572	1.554	1.539	1.527	1.517	1.508	1.500	1.493	1.487	1.481	1.476	1.475	1.467	1.465
0.90	1.725	1.661	1.620	1.590	1.568	1.550	1.536	1.524	1.514	1.506	1.498	1.491	1.485	1.480	1.475	1.472	1.466	1.463
0.95	1.719	1.655	1.615	1.586	1.564	1.547	1.533	1.521	1.511	1.503	1.496	1.489	1.484	1.479	1.474	1.471	1.464	1.461
1.00	1.711	1.650	1.610	1.582	1.560	1.543	1.529	1.518	1.508	1.500	1.492	1.486	1.480	1.475	1.470	1.465	1.462	1.458

and $B(-Q,Y) = -A(Y)$. Utilizing the property $\Phi(-z) = 1 - \Phi(z)$, we have:

$$\begin{aligned}\Phi(B(-Q,Y)) - \Phi(A(-Q,Y)) &= \Phi(-A(Y)) - \Phi(-B(Y)) \\ &= \{1 - \Phi(A(Y))\} - \{1 - \Phi(B(Y))\} \\ &= \Phi(B(Y)) - \Phi(A(Y)) \\ &= \Phi(B(Q,Y)) - \Phi(A(Q,Y))\end{aligned}$$

Thus, the p-values for Q and $(-Q)$ are the same. Hence, the critical value c_0 for Q and $(-Q)$ are also the same for same values of C, n, and α.

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