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The optimal cycle time for EPQ inventory model under permissible delay in payments

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Abstract

Goyal (Journal of the Operational Research Society 36 (1985) 35–38) discusses the economic order quantity under conditions of permissible delay in payments. An implicit assumption of Goyal (Journal of the Operational Research Society 36 (1985) 35–38) is that the items are obtained from an outside supplier. The entire lot size is delivered at the same time. If we wish to adopt all results obtained by Goyal (Journal of the Operational Research Society 36 (1985) 35–38), then we are effectively assuming that the replenishment rate is infinite. The main purpose of this paper is to extend Goyal (Journal of the Operational Research Society 36 (1985) 35–38) to the case that the units are replenished at a finite rate. When the replenishment rate approaches to infinite, Goyal (Journal of the Operational Research Society 36 (1985) 35–38) will be a special case of this paper.

Keywords: EPQ; EOQ; Permissible delay in payments; Trade credit; Inventory

1. Introduction

The EOQ model is widely used by practitioners as a decision-making tool for the control of inventory. The traditional EOQ model assumes that the retailer must be paid for the items as soon as the items are received. However, in practice the supplier will offer the retailer a delay period, that is trade credit period, in paying for the amount of purchasing cost. Before the end of trade credit period, the retailer can sell the goods and accumulate revenue and earn interest. A higher interest is charged if the payment is not settled by the end of trade credit period. In real world, the supplier often makes use of this policy to promote their commodities. Many related articles can be found in Aggarwal and Jaggi (1995), Chang and Dye (2001), Chang et al. (2001), Chen and Chuang (1999), Chu et al. (1998), Chung (2000, 1998a, b), Goyal (1985), Jamal et al. (1997, 2000), Khouja and Mehrez (1996), Liao et al. (2000), Sarker et al. (2000, 2001), and Shah and Shah (1998) and their references.

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Goyal (1985) is the first person to consider the economic order quantity under conditions of permissible delay in payments. Goyal (1985) is frequently cited when the inventory systems under conditions of permissible delay in payments are discussed. An implicit assumption of Goyal (1985) is that the items are obtained from an outside supplier and the entire lot size is delivered at the same time. Therefore, if we wish to adopt all results obtained by Goyal (1985), then we are effectively assuming that the replenishment rate is infinite. When the replenishment rate is much larger than the demand rate, this assumption is probably satisfactory as an approximation. However, if the rate of replenishment is comparable to the rate of demand, Goyal's analysis (1985) needs to be modified to reflect this situation. Consequently, the main purpose of this paper is to extend Goyal's model (1985) to the case that all items are replenished at a finite rate.

2. Model formulation and convexity

The following notation and assumptions will be used throughout:

2.1. Notation

D	demand rate per year
P	replenishment rate per year, $P \geqslant D$
A	cost of placing one order
ρ	$(= 1 - D/P \geqslant 0)$
c	unit purchasing price per item
h	unit stock-holding cost per item per year excluding interest charges
I_e	interest which can be earned per \$ per year
I_k	interest charges per \$ investment in inventory per year
M	permissible delay period
T	the cycle time
$\mathrm{TVC}(T)$	the total relevant cost per unit time when $T > 0$

$$\mathrm{TVC}(T) = \begin{cases} \mathrm{TVC}_1(T) & \text{if } T \geqslant \dfrac{PM}{D}, \\ \mathrm{TVC}_2(T) & \text{if } M \leqslant T \leqslant \dfrac{PM}{D}, \\ \mathrm{TVC}_3(T) & \text{if } T \leqslant M, \end{cases}$$

$$\mathrm{TVC}_1(T) = \frac{A}{T} + \frac{DTh\rho}{2} + cI_k\rho\left(\frac{DT^2}{2} - \frac{PM^2}{2}\right)/T - cI_e\left(\frac{DM^2}{2}\right)/T \quad \text{if } T > 0,$$

$$\mathrm{TVC}_2(T) = \frac{A}{T} + \frac{DTh\rho}{2} + cI_k\left[\frac{D(T-M)^2}{2}\right]/T - cI_e\left(\frac{DM^2}{2}\right)/T \quad \text{if } T > 0,$$

$$\mathrm{TVC}_3(T) = \frac{A}{T} + \frac{DTh\rho}{2} - cI_e\left[\frac{DT^2}{2} + DT(M-T)\right]/T \quad \text{if } T > 0,$$

$$T_1^* = \sqrt{\frac{2A + DM^2c(I_k - I_e) - PM^2cI_k}{D\rho(h + cI_k)}} \quad \text{if } 2A + DM^2c(I_k - I_e) - PM^2cI_k > 0,$$

$$T_2^* = \sqrt{\frac{2A + DM^2c(I_k - I_e)}{D(h\rho + cI_k)}},$$

$$T_3^* = \sqrt{\frac{2A}{D(h\rho + cI_e)}},$$

T^* the optimal cycle time of TVC(T).

2.2. Assumptions

(1) Demand rate, D, is known and constant.
(2) Replenishment rate, P, is known and constant.
(3) Shortages are not allowed.
(4) Time period is infinite.
(5) $I_k \geqslant I_e$.
(6) During the time the account is not settled, generated sales revenue is deposited in an interest-bearing account. When $T \geqslant M$, the account is settled at $T = M$ and we start paying for the interest charges on the items in stock. When $T \leqslant M$, the account is settled at $T = M$ and we do not need to pay any interest charge.

The annual total relevant cost consists of the following elements.

(1) Annual ordering cost $= (A/T)$.
(2) Annual stock-holding cost (excluding interest charges) (shown in Fig. 1)

$$= \frac{hT(P - D)(DT/P)}{2T} = \frac{DTh}{2}\left(1 - \frac{D}{P}\right) = \frac{DTh\rho}{2}.$$

(3) There are three cases to occur in costs of interest charges for the items kept in stock per year.

Case 1: $M \leqslant PM/D \leqslant T$, shown in Fig. 1.

$$\text{Annual interest payable} = cI_k\left[\frac{DT^2\rho}{2} - \frac{(P - D)M^2}{2}\right]/T = cI_k\rho\left(\frac{DT^2}{2} - \frac{PM^2}{2}\right)/T. \quad (1)$$

Case 2: $M \leqslant T \leqslant PM/D$, shown in Fig. 2.

$$\text{Annual interest payable} = cI_k\left[\frac{D(T - M)^2}{2}\right]/T. \quad (2)$$

Case 3: $T \leqslant M$.
In this case, no interest charges are paid for the items.

(4) There are three cases to occur in interest earned per year.

Case 1: $M \leqslant PM/D \leqslant T$.

$$\text{Annual interest earned} = cI_e\left(\frac{DM^2}{2}\right)/T. \quad (3)$$

Case 2: $M \leqslant T \leqslant PM/D$.

$$\text{Annual interest earned} = cI_e\left(\frac{DM^2}{2}\right)/T. \quad (4)$$

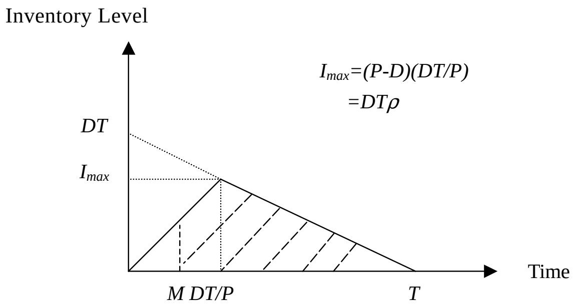

Fig. 1. The total accumulation of interest payable when $PM/D \leqslant T$.

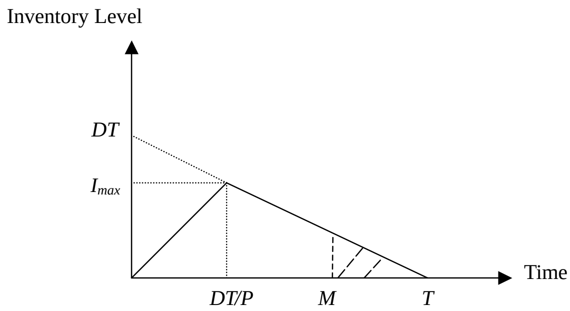

Fig. 2. The total accumulation of interest payable when $M \leqslant T \leqslant PM/D$.

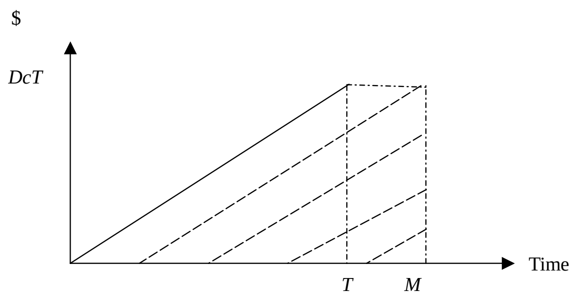

Fig. 3. The total accumulation of interest earned when $T \leqslant M$.

Case 3: $T \leqslant M$, shown in Fig. 3.

$$\text{Annual interest earned} = cI_e\left[\frac{DT^2}{2} + DT(M-T)\right]/T. \tag{5}$$

From the above arguments, the annual total relevant cost for the retailer can be expressed as TVC(T) = ordering cost + stock-holding cost + interest payable − interest earned.

We show that the annual total relevant cost, TVC(T), is given by

$$\mathrm{TVC}(T) = \begin{cases} \mathrm{TVC}_1(T) & \text{if } T \geqslant \dfrac{PM}{D}, \\ \mathrm{TVC}_2(T) & \text{if } M \leqslant T \leqslant \dfrac{PM}{D}, \\ \mathrm{TVC}_3(T) & \text{if } 0 < T \leqslant M, \end{cases} \tag{6a–c}$$

where

$$\mathrm{TVC}_1(T) = \frac{A}{T} + \frac{DTh\rho}{2} + cI_k\rho\left(\frac{DT^2}{2} - \frac{PM^2}{2}\right)/T - cI_e\left(\frac{DM^2}{2}\right)/T, \tag{7}$$

$$\mathrm{TVC}_2(T) = \frac{A}{T} + \frac{DTh\rho}{2} + cI_k\left[\frac{D(T-M)^2}{2}\right]/T - cI_e\left(\frac{DM^2}{2}\right)/T, \tag{8}$$

$$\mathrm{TVC}_3(T) = \frac{A}{T} + \frac{DTh\rho}{2} - cI_e\left[\frac{DT^2}{2} + DT(M-T)\right]/T. \tag{9}$$

Since $\mathrm{TVC}_1(PM/D) = \mathrm{TVC}_2(PM/D)$ and $\mathrm{TVC}_2(M) = \mathrm{TVC}_3(M)$, $\mathrm{TVC}(T)$ is continuous and well defined. All $\mathrm{TVC}_1(T)$, $\mathrm{TVC}_2(T)$, $\mathrm{TVC}_3(T)$ and $\mathrm{TVC}(T)$ are defined on $T>0$. Eqs. (7), (8) and (9) yield

$$\mathrm{TVC}_1'(T) = -\left[\frac{2A - M^2(cI_kP\rho + DcI_e)}{2T^2}\right] + D\rho\left(\frac{h+cI_k}{2}\right) = -\left[\frac{2A + DM^2c(I_k - I_e) - PM^2cI_k}{2T^2}\right] + D\rho\left(\frac{h+cI_k}{2}\right), \tag{10}$$

$$\mathrm{TVC}_1''(T) = \frac{2A - M^2(cI_kP\rho + DcI_e)}{T^3} = \frac{2A + DM^2c(I_k - I_e) - PM^2cI_k}{T^3}. \tag{11}$$

$$\mathrm{TVC}_2'(T) = -\left[\frac{2A + DM^2c(I_k - I_e)}{2T^2}\right] + D\left(\frac{h\rho + cI_k}{2}\right), \tag{12}$$

$$\mathrm{TVC}_2''(T) = \frac{2A + DM^2c(I_k - I_e)}{T^3} > 0, \tag{13}$$

$$\mathrm{TVC}_3'(T) = \frac{-A}{T^2} + D\left(\frac{h\rho + cI_e}{2}\right) \tag{14}$$

and

$$\mathrm{TVC}_3''(T) = \frac{2A}{T^3} > 0. \tag{15}$$

Eqs. (13) and (15) imply that $\mathrm{TVC}_2(T)$ and $\mathrm{TVC}_3(T)$ are convex on $T>0$. However, $\mathrm{TVC}_1(T)$ is convex on $T>0$ if $2A + DM^2c(I_k - I_e) - PM^2cI_k > 0$. Furthermore, we have $\mathrm{TVC}_1'(PM/D) = \mathrm{TVC}_2'(PM/D)$ and $\mathrm{TVC}_2'(M) = \mathrm{TVC}_3'(M)$. Therefore, Eqs. (6a–c) imply that $\mathrm{TVC}(T)$ is convex on $T>0$ if $2A + DM^2c(I_k - I_e) - PM^2cI_k > 0$. Since Eqs. (11), (13) and (15), $\mathrm{TVC}_1'(PM/D) = \mathrm{TVC}_2'(PM/D)$ and $\mathrm{TVC}_2'(M) = \mathrm{TVC}_3'(M)$, we have the following results:

Theorem 1. (A) *If* $2A + DM^2c(I_k - I_e) - PM^2cI_k \leqslant 0$, *then* $\mathrm{TVC}(T)$ *is convex on* $(0, PM/D]$ *and concave on* $[PM/D, \infty)$.

(B) *If* $2A + DM^2c(I_k - I_e) - PM^2cI_k > 0$, *then* $\mathrm{TVC}(T)$ *is convex on* $(0, \infty)$.

3. The determination of the optimal cycle time T^*

Recall

$$T_1^* = \sqrt{\frac{2A + DM^2c(I_k - I_e) - PM^2cI_k}{D\rho(h + cI_k)}} \quad \text{if } 2A + DM^2c(I_k - I_e) - PM^2cI_k > 0, \tag{16}$$

$$T_2^* = \sqrt{\frac{2A + DM^2c(I_k - I_e)}{D(h\rho + cI_k)}} \tag{17}$$

and

$$T_3^* = \sqrt{\frac{2A}{D(h\rho + cI_e)}} \tag{18}$$

introduced in the previous section. Then $\text{TVC}_i'(T_i^*) = 0$ for all $i = 1, 2, 3$. Furthermore, we have the following results:

Lemma 1. *If* $2A + DM^2c(I_k - I_e) - PM^2cI_k \leqslant 0$, *then* $T_2^* < (PM/D)$.

Proof. If $T_2^* \geqslant (PM/D)$, then

$$\frac{2A + DM^2c(I_k - I_e)}{D(h\rho + cI_k)} \geqslant \frac{P^2M^2}{D^2}. \tag{19}$$

Eq. (19) yields

$$2A + DM^2c(I_k - I_e) \geqslant \frac{P^2M^2}{D}(h\rho + cI_k). \tag{20}$$

Therefore, we have

$$2A + DM^2c(I_k - I_e) - PM^2cI_k \geqslant \frac{PM^2}{D}[Ph\rho + cI_k(P - D)] > 0. \tag{21}$$

Eq. (21) is a contradiction. Consequently, $T_2^* < (PM/D)$. □

Lemma 2. $T_3^* \leqslant M$ *if and only if* $T_2^* \leqslant M$.

Proof. If $T_3^* \leqslant M$, Eq. (18) implies

$$2A \leqslant DM^2(h\rho + cI_e).$$

Hence

$$2A + DM^2c(I_k - I_e) \leqslant DM^2(h\rho + cI_e) + DM^2c(I_k - I_e).$$

We have

$$\sqrt{\frac{2A + DM^2c(I_k - I_e)}{D(h\rho + cI_k)}} \leqslant M$$

and

$$T_2^* \leqslant M.$$

Similarly, if $T_2^* \leqslant M$, we can obtain $T_3^* \leqslant M$. Combining the above arguments, this completes the proof of Lemma 2. □

Lemmas 1 and 2 imply the following theorem:

Theorem 2. *Suppose that* $2A + DM^2c(I_k - I_e) - PM^2cI_k \leqslant 0$. *Then*

(A) *If* $T_3^* < M$, *then* $T^* = T_3^*$.
(B) *If* $T_3^* \geqslant M$, *then* $T^* = T_2^*$.

Proof. If $2A + DM^2c(I_k - I_e) - PM^2cI_k \leqslant 0$, Eq. (10) implies that $\mathrm{TVC}_1(T)$ is increasing on $[PM/D, \infty)$. There are two cases which occur:

(A) Suppose that $T_3^* < M$. Lemma 2 implies $T_2^* < M$. Hence

(i) $\mathrm{TVC}_2(T)$ is increasing on $[M, PM/D]$.
(ii) $\mathrm{TVC}_3(T)$ is decreasing on $(0, T_3^*]$ and increasing on $[T_3^*, M]$.

Combining (i), (ii) and Eqs. (6a–c), we have that $\mathrm{TVC}(T)$ is decreasing on $(0, T_3^*]$ and increasing on $[T_3^*, \infty)$. Consequently, $T^* = T_3^*$.

(B) Suppose that $T_3^* \geqslant M$. Lemmas 1 and 2 imply $M \leqslant T_2^* \leqslant PM/D$. Hence

(i) $\mathrm{TVC}_2(T)$ is decreasing on $[M, T_2^*]$ and increasing on $[T_2^*, PM/D]$.
(ii) $\mathrm{TVC}_3(T)$ is decreasing on $(0, M]$.

Combining (i), (ii) and Eqs. (6a–c), we have that $\mathrm{TVC}(T)$ is decreasing on $(0, T_2^*]$ and increasing on $[T_2^*, \infty)$. Consequently, $T^* = T_2^*$.

Incorporating the above arguments, we have completed the proof of Theorem 2. □

If $2A + DM^2c(I_k - I_e) - PM^2cI_k > 0$, then $\mathrm{TVC}_i(T)$ is convex for all $i = 1, 2, 3$. By the convexity of $\mathrm{TVC}_i(T)$ $(i = 1, 2, 3)$, we see

$$\mathrm{TVC}_i'(T) \begin{cases} < 0 & \text{if } T < T_i^* \\ = 0 & \text{if } T = T_i^*, \\ > 0 & \text{if } T > T_i^*. \end{cases} \tag{22a-c}$$

Eqs. (22a–c) imply that $\mathrm{TVC}_i(T)$ is decreasing on $(0, T_i^*]$ and increasing on $[T_i^*, \infty)$ for all $i = 1, 2, 3$. Eqs. (10), (12) and (14) yield

$$\mathrm{TVC}_1'\left(\frac{PM}{D}\right) = \mathrm{TVC}_2'\left(\frac{PM}{D}\right) = \frac{-2A + (M^2/D)[P(P-D)h + cI_k(P^2 - D^2) + cI_eD^2]}{2(PM/D)^2} \tag{23}$$

and

$$\mathrm{TVC}_2'(M) = \mathrm{TVC}_3'(M) = \frac{-2A + DM^2(h\rho + cI_e)}{2M^2}. \tag{24}$$

Furthermore, we let

$$\Delta_1 = -2A + \frac{M^2}{D}[P(P-D)h + cI_k(P^2 - D^2) + cI_eD^2]$$

and

$$\Delta_2 = -2A + DM^2(h\rho + cI_e).$$

Then, we have

$$\Delta_1 \geqslant \Delta_2, \tag{25}$$

$$\Delta_1 > 0 \quad \text{if and only if } T_1^* < PM/D, \tag{26}$$

$$\Delta_2 > 0 \quad \text{if and only if } T_3^* < M. \tag{27}$$

Lemma 3. *$T_1^* \leqslant PM/D$ if and only if $T_2^* \leqslant PM/D$.*

Proof. If $T_1^* \leqslant PM/D$, Eq. (16) implies

$$2A + DM^2c(I_k - I_e) - PM^2cI_k \leqslant (P^2M^2/D^2)[D\rho(h + cI_k)].$$

Hence

$$2A + DM^2c(I_k - I_e) \leqslant (P^2M^2/D^2)[D(h\rho + cI_k)].$$

We have

$$\sqrt{\frac{2A + DM^2c(I_k - I_e)}{D(h\rho + cI_k)}} \leqslant \frac{PM}{D}$$

and

$$T_2^* \leqslant PM/D.$$

Similarly, if $T_2^* \leqslant PM/D$, we can obtain $T_1^* \leqslant PM/D$. Combining the above arguments, this completes the proof of Lemma 3. □

Combining above results (i), (ii), (iii) and Lemma 3, we have the following theorem:

Theorem 3. *Suppose that $2A + DM^2c(I_k - I_e) - PM^2cI_k > 0$. Then*

(A) *If $\Delta_2 \geqslant 0$ and $\Delta_1 > 0$, then* $\mathrm{TVC}(T^*) = \mathrm{TVC}(T_3^*)$ *and* $T^* = T_3^*$.
(B) *If $\Delta_1 \leqslant 0$ and $\Delta_2 < 0$, then* $\mathrm{TVC}(T^*) = \mathrm{TVC}(T_1^*)$ *and* $T^* = T_1^*$.
(C) *If $\Delta_1 > 0$ and $\Delta_2 < 0$, then* $\mathrm{TVC}(T^*) = \mathrm{TVC}(T_2^*)$ *and* $T^* = T_2^*$.

Proof. (A) If $\Delta_2 \geqslant 0$ and $\Delta_1 > 0$, then $T_3^* \leqslant M, T_2^* \leqslant M, T_1^* < PM/D$ and $T_2^* < PM/D$. We have $\mathrm{TVC}_1'(PM/D) = \mathrm{TVC}_2'(PM/D) > 0$ and $\mathrm{TVC}_2'(M) = \mathrm{TVC}_3'(M) \geqslant 0$. Eqs. (22a–c) imply that

(i) $\mathrm{TVC}_1(T)$ is increasing on $[PM/D, \infty)$.
(ii) $\mathrm{TVC}_2(T)$ is increasing on $[M, PM/D]$.
(iii) $\mathrm{TVC}_3(T)$ is decreasing on $(0, T_3^*]$ and increasing on $[T_3^*, M]$.

Combining (i), (ii), (iii) and Eqs. (6a–c), we have that TVC(T) is decreasing on (0, T_3^*] and increasing on [T_3^*, ∞). Consequently, $T^* = T_3^*$.

(B) If $\Delta_1 \leqslant 0$ and $\Delta_2 < 0$, then $T_3^* > M, T_2^* > M, T_1^* \geqslant PM/D$ and $T_2^* \geqslant PM/D$. We have $\text{TVC}_1'(PM/D) = \text{TVC}_2'(PM/D) \leqslant 0$ and $\text{TVC}_2'(M) = \text{TVC}_3'(M) < 0$. Eqs. (22a–c) imply that

(i) $\text{TVC}_1(T)$ is decreasing on [(PM/D), T_1^*] and $\text{TVC}_1(T)$ is increasing on [T_1^*, ∞).
(ii) $\text{TVC}_2(T)$ is decreasing on [M, (PM/D)].
(iii) $\text{TVC}_3(T)$ is decreasing on (0, M].

Combining (i), (ii), (iii) and Eqs. (6a–c), we have that TVC(T) is decreasing on (0, T_1^*] and increasing on [T_1^*, ∞). Consequently, $T^* = T_1^*$.

(C) If $\Delta_1 > 0$ and $\Delta_2 < 0$, then $T_3^* > M, T_2^* > M$, $T_1^* < PM/D$ and $T_2^* < PM/D$. We have $\text{TVC}_1'(PM/D) = \text{TVC}_2'(PM/D) > 0$ and $\text{TVC}_2'(M) = \text{TVC}_3'(M) < 0$. Eqs. (22a–c) imply that

(i) $\text{TVC}_1(T)$ is increasing on [PM/D, ∞).
(ii) $\text{TVC}_2(T)$ is decreasing on [M, T_2^*] and $\text{TVC}_2(T)$ is increasing on [T_2^*, PM/D].
(iii) $\text{TVC}_3(T)$ is decreasing on (0, M].

Combining (i), (ii), (iii) and Eqs. (6a–c), we have that TVC(T) is decreasing on (0, T_2^*] and increasing on [T_2^*, ∞). Consequently, $T^* = T_2^*$.

Incorporating the above arguments, we have completed the proof of Theorem 3. □

4. Special case

When $P \to \infty$, then

$$\lim_{P\to\infty} \text{TVC}_2(T) = \frac{A}{T} + \frac{DTh}{2} + cI_k\left[\frac{D(T-M)^2}{2}\right]/T - cI_e\left(\frac{DM^2}{2}\right)/T,$$

$$\lim_{P\to\infty} \text{TVC}_3(T) = \frac{A}{T} + \frac{DTh}{2} - cI_e\left[\frac{DT^2}{2} + DT(M-T)\right]/T,$$

$$\lim_{P\to\infty} T_2^* = \sqrt{\frac{2A + DM^2c(I_k - I_e)}{D(h + cI_k)}}$$

and

$$\lim_{P\to\infty} T_3^* = \sqrt{\frac{2A}{D(h + cI_e)}}.$$

Let

$$\text{TVC}_4(T) = \frac{A}{T} + \frac{DTh}{2} + cI_k\left[\frac{D(T-M)^2}{2}\right]/T - cI_e\left(\frac{DM^2}{2}\right)/T,$$

$$\text{TVC}_5(T) = \frac{A}{T} + \frac{DTh}{2} - cI_e\left[\frac{DT^2}{2} + DT(M-T)\right]/T,$$

$$\bar{T}_2^* = \sqrt{\frac{2A + DM^2c(I_k - I_e)}{D(h + cI_k)}}$$

and

$$\bar{T}_3^* = \sqrt{\frac{2A}{D(h + cI_e)}}.$$

Then, Eqs. (6a–c) will be reduced as follows:

$$\mathrm{TVC}(T) = \begin{cases} \mathrm{TVC}_4(T) & \text{if } M \leqslant T, \\ \mathrm{TVC}_5(T) & \text{if } 0 < T \leqslant M. \end{cases} \tag{28a, b}$$

Eqs. (28a, b) will be consistent with Eqs. (1) and (4) in Goyal (1985), respectively. Hence, Goyal (1985) will be a special case of this paper.

Since

$$\lim_{P \to \infty} \Delta_1 = \infty$$

and

$$\lim_{P \to \infty} \Delta_2 = -2A + DM^2(h + cI_e),$$

if we let $\Delta = -2A + DM^2(h + cI_e)$, Theorem 3 can be modified as follows:

Theorem 4. (A) *If* $\Delta > 0$, *then* $T^* = \bar{T}_3^*$.
(B) *If* $\Delta < 0$, *then* $T^* = \bar{T}_2^*$.
(C) *If* $\Delta = 0$, *then* $T^* = \bar{T}_2^* = \bar{T}_3^* = M$.

Theorem 4 has been discussed in Chung (1998a). Hence, Theorem 1 in Chung (1998a) is a special case of Theorem 3 of this paper.

5. Numerical examples

To illustrate the results, let us apply the proposed method to solve the following numerical examples:

Example 1. Let $A = \$250/\text{order}$, $D = 4000$ units/year, $P = 5000$ units/year, $M = 0.1$ year, $c = \$100/\text{unit}$, $I_k = \$0.15/\$/\text{year}$, $I_e = \$0.12/\$/\text{year}$, $h = \$5/\text{unit/year}$. Therefore, $2A + DM^2c(I_k - I_e) - PM^2cI_k = -130 < 0$ and $T_3^* = 0.09806 < M = 0.1$ year. Using Theorem 2(A), we get $T^* = T_3^* = 0.09806$ year. The optimal order quantity will be $DT_3^* = 392$ units. $\mathrm{TVC}(T^*) = \mathrm{TVC}(T_3^*)$=\$299.

Example 2. Let $A = \$100/\text{order}$, $D = 2000$ units/year, $P = 3000$ units/year, $M = 0.1$ year, $c = \$60/\text{unit}$, $I_k = \$0.15/\$/\text{year}$, $I_e = \$0.12/\$/\text{year}$, $h = \$5/\text{unit/year}$. Therefore, $2A + DM^2c(I_k - I_e) - PM^2cI_k = -34 < 0$ and $T_3^* = 0.1062 > M = 0.1$ year. Using Theorem 2(B), we get $T^* = T_2^* = 0.1052$ year. The optimal order quantity will be $DT_2^* = 210$ units. $\mathrm{TVC}(T^*) = \mathrm{TVC}(T_2^*)$=\$441.5.

Example 3. Let $A = \$100/\text{order}$, $D = 2600$ units/year, $P = 3000$ units/year, $M = 0.1$ year, $c = \$50/\text{unit}$, $I_k = \$0.15/\$/\text{year}$, $I_e = \$0.13/\$/\text{year}$, $h = \$10/\text{unit/year}$. Therefore, $2A + DM^2c(I_k - I_e) - PM^2cI_k = 1 > 0$, $\Delta_1 = 79.8 > 0$ and $\Delta_2 = 3.67 > 0$. Using Theorem 3(A), we get $T^* = T_3^* = 0.0991$ year. The optimal order quantity will be $DT_3^* = 258$ units. $\mathrm{TVC}(T^*) = \mathrm{TVC}(T_3^*)$=\$328.3.

Example 4. Let $A = \$100/\text{order}$, $D = 2500$ units/year, $P = 3000$ units/year, $M = 0.1$ year, $c = \$35/\text{unit}$, $I_k = \$0.15/\$/\text{year}$, $I_e = \$0.12/\$/\text{year}$, $h = \$5/\text{unit/year}$. Therefore, $2A + DM^2c(I_k - I_e) - PM^2cI_k = 68.8 > 0$, $\Delta_1 = -7.25 < 0$ and $\Delta_2 = -74.17 < 0$. Using Theorem 3(B), we get $T^* = T_1^* = 0.1269$ year. The optimal order quantity will be $DT_1^* = 317$ units. $\text{TVC}(T^*) = \text{TVC}(T_1^*) = \$541.9$.

Example 5. Let $A = \$100/\text{order}$, $D = 3000$ units/year, $P = 3200$ units/year, $M = 0.1$ year, $c = \$50/\text{unit}$, $I_k = \$0.15/\$/\text{year}$, $I_e = \$0.12/\$/\text{year}$, $h = \$5/\text{unit/year}$. Therefore, $2A + DM^2c(I_k - I_e) - PM^2cI_k = 5 > 0$, $\Delta_1 = 21.7 > 0$ and $\Delta_2 = -10.6 < 0$. Using Theorem 3(C), we get $T^* = T_2^* = 0.1022$ year. The optimal order quantity will be $DT_2^* = 307$ units. $\text{TVC}(T^*) = \text{TVC}(T_2^*)$=\$145.7.

6. Conclusions

This paper extends Goyal (1985) to the case that the units are replenished at a finite rate. Theorem 1 explores the convexity of the annual total relevant cost function. On the other hand, Theorems 2 and 3 describe the effective solution procedure to find the optimal cycle time of the annual total relevant cost function. If the replenishment rate approaches to infinite, the inventory model discussed in this paper is reduced to Goyal (1985). Consequently, Goyal (1985) is a special case of this paper. Finally, numerical examples are used to illustrate all results obtained by this paper.

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