字首質式之一般表示式的特徵

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摘要

質式字之基本組合性質在正規言語理論中扮演著相當重要的角色。本文探討類似於質式字的字首質式字。字首質式的性質被應用於檢核一組資料在設計一類神經網路時是否造成收斂。本文整理出兩個不同的字 u 與 v 之特徵，其中 u 的字長小於 v 的字長，使得 uv 的一般表示式為字首質式。

關鍵字：質式字；字首質式字

A Note of P-Primitive Regular Expressions

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Abstract: Primitive words play a very important role in formal language theory for their elementary combinatorial properties. Analogous to primitive words, we consider p-primitive words. P-primitivities are applied to check whether a neural network converges for a set of data. In this note we characterize p-primitive words u and words v, where $\lg(u) < \lg(v)$, such that the regular expression uv^+ is p-primitive.

Keywords: Primitive word; p-primitive word

1. Introduction

Let X be an *alphabet* which contains more than one letter. Let X^* be the *free monoid generated by* X and $X^+ = X^* \setminus \{1\}$ where 1 is the *empty word*. For a word $u \in X^*$, let $\lg(u)$ denote the length of u. For $u, v \in X^+$, u is called a *power* of v if $u = v^n$ for some integer $n \geq 1$. A nonempty word u is called *primitive* if u is not a power of any other word. It is known that every word $u \in X^+$ is a power of a unique primitive word ([1]). If $u = xy, x, y \in X^*$, then x is called a *prefix* of u, denoted by $x \leq_{\mathrm{p}} u$. If $x \neq u$, then x is said to be a *proper prefix* of u, denoted by $x <_{\mathrm{p}} u$. A word w has a *prefix n-power* if $w \in u^n X^*$ for some $u \in X^+$. For $w \in X^+$, let $N(w)$ denote the maximal number n such that w has a prefix n-power. For any $i \geq 1$, we define $P_i(X)$ as $P_i(X) = \{w \in X^+ \mid N(w) = i\}$. From the definition, it is clear that $P_i(X) \cap P_j(X) = \emptyset$ for every $i \neq j$ and $X^+ = \bigcup_{i \geq 1} P_i(X)$. Every word w in $P_1(X)$ is called *prefix primitive* (shortly, *p-primitive*), i.e., $w \notin u^2 X^*$ for any $u \in X^+$. Let $X = \{a, b\}$. Then ab^n is a p-primitive word over X for any $n \geq 1$.

In this paper we investigate that for any two distinct words u and v, where $u \in P_1(X)$ and $\lg(u) < \lg(v)$, whether or not uv^+ is p-primitive. A language in this form uv^+w for some $u, v, w \in X^*$ is called a *regular component* ([3]). In section 2, Proposition 2.1 to Proposition 2.3 are concerned that some characters of words u and v which lead uv^n is not a p-primitive word. On the contrary, if uv^n is not a p-primitive word, then u and v must be those character of words. Proposition 2.4 show that if $uv^n \in P_1(X)$, for $n \leq 3$, then $uv^n \in P_1(X)$ for all $n \geq 4$.

The following two lemmata concerning the basic properties of the catenation and decompositions of words will be needed in the sequel.

Lemma 1.1 ([1]) If $uv = vu$, $u, v \in X^+$, then u and v are powers of a common word.

Lemma 1.2 ([1]) If $uv = vz$, $u, v, z \in X^*$ and $u \neq 1$, then $u = xy, v = (xy)^k x, z = yx$ for some $x, y \in X^*$ and $k \geq 0$.

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2. Main Results

It is known that if $uv^i \notin P_1(X)$, then $uv^j \notin P_1(X)$, for all $j > i$. Therefore, if we are going to discuss whether or not uv^j is a p-primitive word, we first assume that uv^i is a p-primitive word, for all $i < j$. Now, we give a characterization of words uv^n being p-primitive for all $n \leq 3$.

Proposition 2.1 For any two distinct words u, v, where $u \in P_1(X)$ and $\lg(u) < \lg(v)$. Then uv is not a p-primitive word if and only if one of the following three statements holds:
(1) $u <_p v$,
(2) $u = x_1x_2x_1$ for some $x_1, x_2 \in X^+$ with $x_2 <_p v$,
(3) $v = x_1ux_1x_2$ for some $x_1, x_2 \in X^+$.

Proof. ($\Rightarrow$) As uv is not a p-primitive word, $uv = x^2y$ for some $x \in X^+$ and $y \in X^*$. As $uv = x^2$, $\lg(u) < \lg(x)$. There exists $x_1 \in X^+$ such that $x = ux_1$. Thus $v = x_1x = x_1ux_1$. The assertion with statement (3) holds, where x_2 is an empty word. As $uv = x^2y$ for some $x, y \in X^+$, if $\lg(y) \geq \lg(v)$, then $x^2 \leq_p u$. This leads to a contradiction. Hence $\lg(y) < \lg(v)$. Let $v = v_1y$, where $v_1, y \in X^+$. As $uv = uv_1y = x^2y$, we get $uv_1 = x^2$. If $\lg(u) = \lg(v_1)$, then $u = v_1 = x$. This yileds $u <_p v$. The assertion with statement (1) holds. If $\lg(u) < \lg(v_1)$, then $\lg(u) < \lg(x)$. There exists $x_1 \in X^+$ such that $x = ux_1$. This yields $v_1 = x_1x = x_1ux_1$, $v = x_1ux_1y$. The assertion with statement (3) holds, where $x_2 = y$. If $\lg(u) > \lg(v_1)$, then $\lg(u) > \lg(x)$. There exists $x_1 \in X^+$ such that $u = xx_1$. Then $x = x_1x_2$, where $x_2 \in X^+$. We get $u = x_1x_2x_1$. Since $uv = x^2y = (x_1x_2)^2y = ux_2y$. This yields $x_2 <_p v$. The assertion with statement (2) holds.

($\Leftarrow$) Immediate. ∎

Proposition 2.2 For any two distinct words u, v, where $u, uv \in P_1(X)$ and $\lg(u) < \lg(v)$. Then uv^2 is not a p-primitive word if and only if one of the following statements holds:
(1) $u = x_1x_2x_3$ and $v = x_3x_4x_1x_2$ for some $x_1, x_3, x_4 \in X^+$ and $x_2 \in X^*$ with $x_3x_4 = x_4x_1$,
(2) $v = x_1x_2x_3ux_1$ for some $x_1, x_3 \in X^+$ and $x_2 \in X^*$ with $x_1x_2 = x_2x_3$,
(3) $v = x_1x_2x_1ux_1x_2$ for some $x_1 \in X^+$ and $x_2 \in X^*$,
(4) $v = x_1u$ for some $x_1 \in X^+$.

Proof. ($\Rightarrow$) As uv^2 is not a p-primitive word, $uv^2 = x^2y$ for some $x \in X^+$ and $y \in X^*$. As $uv^2 = x^2$, $\lg(x) < \lg(uv)$. There exist $v_1, v_2 \in X^+$ such that $x = uv_1 = v_2v_1v_2$, where $v = v_1v_2$. Thus $\lg(u) = 2\lg(v_2)$. As $v_2 <_p u$, there exists $u_1 \in X^+$ such that $u = v_2u_1$. This yields $\lg(u_1) = \lg(v_2)$. Since $\lg(u) < \lg(v)$, $\lg(u_1) < \lg(v_1)$. This yields $u_1 <_p v_1$. Then there exists $v_3 \in X^+$ such that $v_1 = u_1v_3$. As $x = v_2u_1v_1 = v_2v_1v_2$, $v_2u_1u_1v_3 = v_2u_1v_3v_2$. This implies that $u_1v_3 = v_3v_2$. Hence $u = v_2u_1$, $v = u_1v_3v_2$ and $u_1v_3 = v_3v_2$. The assertion with statement (1) holds, where $x_1 = v_2, x_3 = u_1, x_4 = v_3$ and x_2 is an empty word. As $uv^2 = x^2y$ for some $x, y \in X^+$, if $\lg(y) \geq \lg(v)$, then $x^2 \leq_p uv$. This leads to a contradiction. Hence $\lg(y) < \lg(v)$. Let $v = v_1y$, where $v_1, y \in X^+$. Then $x^2 = uv_1yv_1$. Consider the following cases:

(1) $\lg(x) = \lg(uv_1)$. Then $u = y$ and $v = v_1u$. The assertion with statement (4) holds, where $x_1 = v_1$.

(2) $\lg(x) < \lg(uv_1)$. There exist $v_2, v_3 \in X^+$ such that $v_1 = v_2v_3$ and $x = uv_2 = v_3yv_2v_3$. If $\lg(v_2) = \lg(v_3)$, then $v_2 = v_3$. We get $u = v_2yv_2$ and $v = v_2v_2y$. Hence $\lg(u) = \lg(v)$. This contradicts the fact that $\lg(u) < \lg(v)$. If $\lg(v_2) < \lg(v_3)$, then there exists $v_4 \in X^+$ such that $v_3 = v_4v_2$. We can get $u = v_3yv_2v_4$ and $v = v_2v_3y$. Hence $\lg(u) > \lg(v)$. This also contradicts the fact that $\lg(u) < \lg(v)$. Then $\lg(v_2) > \lg(v_3)$. As $v_2 <_s x$ and $v_3 <_s x$. Then there exist $v_4, v_5 \in X^+$ such that $v_2 = v_4v_5 = v_5v_3$, we get $u = v_3yv_4$ and $v = v_4v_5v_3y$. The assertion with statement (1) holds, where $x_1 = v_3$, $x_2 = y$, $x_3 = v_4$, $x_4 = v_5$.

(3) $\lg(x) > \lg(uv_1)$. There exist $v_2, v_3 \in X^+$ such that $y = v_2v_3$ and $x = uv_1v_2 = v_3v_1$. Thus $\lg(u) < \lg(v_3)$ and $v = v_1v_2v_3$. If $\lg(v_1) = \lg(v_2)$, then $v_1 = v_2$. As $x = uv_1v_2 = uv_1v_1 = v_3v_1$, we get $v_3 = uv_1$. Thus $v = v_1v_2v_3 = v_1v_1uv_1$. The assertion with statement (2) holds, where $x_1 = x_3 = v_1$ and x_2 is an empty word. If $\lg(v_1) > \lg(v_2)$, then there exist $v_4, v_5 \in X^+$ such that $v_1 = v_4v_2 = v_5v_4$. As $x = uv_1v_2 = v_3v_1$, $uv_4v_2v_2 = v_3v_4v_2$. As $\lg(u) < \lg(v_3)$, $v_3 = uv_5$. Hence $v = v_1v_2v_3 = v_4v_2v_2uv_5 = v_5v_4v_2uv_5 = v_5v_5v_4uv_5$. The assertion with statement (2) holds, where $x_1 = v_5, x_2 = v_4$ and $x_3 = v_2$. If $\lg(v_1) < \lg(v_2)$, then there exists $v_4 \in X^+$ such that $v_2 = v_4v_1$. Thus $x = uv_1v_2 = uv_1v_4v_1 = v_3v_1$. This yields $v_3 = uv_1v_4$. Hence $v = v_1v_2v_3 = v_1v_4v_1uv_1v_4$. The assertion with statement (3) holds, where $x_1 = v_1, x_2 = v_4$.

($\Leftarrow$) Immediate. ∎

Proposition 2.3 For any two distinct words u, v, where $u, uv, uv^2 \in P_1(X)$ and $\lg(u) < \lg(v)$. Then uv^3 is not a p-primitive word if and only if one of the following statements holds:

(1) $u = x_1x_2x_2x_3x_1$ and $v = (x_2x_3x_1)^kx_2$ for some $x_1, x_2, x_3 \in X^+$ and $k = 2, 3, 4$ with $x_3 \leq_p x_2$,

(2) $u = x_1x_2x_1x_3$ and $v = (x_2x_1x_3)^2x_2x_1$ for some $x_1, x_2, x_3 \in X^+$ with $x_3x_2 <_p x_2x_1x_3$,

(3) $u = x_2x_4x_3x_4x_1$ and $v = (x_3x_4x_1)^kx_3x_4$ for some $x_1, x_2, x_3 \in X^+$ and $x_4 \in X^*$ with $x_1 = w_1w_2$, $x_2 = w_2w_1$ and $x_3 = (w_1w_2)^{n+i}w_1$, where $w_1, w_2, w_3 \in X^+$, $i = 1, 2$ and $k = 2, 3, 4$,

(4) $u = x_3x_1x_2x_3$ and $v = (x_2x_3)^kx_1$ for some $x_1, x_2 \in X^+$ and $x_3 \in X^*$ with $x_1 = w_1w_2$ and $x_2 = (w_1w_2)^{n+i}w_1$, where $w_1, w_2, w_3 \in X^+$, $i = 1, 2$ and $k = 2, 3, 4$.

Proof. ($\Rightarrow$) As uv^3 is not a p-primitive word, $uv^3 = x^2y$ for some $x \in X^+$ and $y \in X^*$. As $uv^3 = x^2$, $\lg(uv) < \lg(x)$. There exist $v_1, v_2 \in X^+$ such that $x = uv_1v_2v_1 = v_2v_1v_2$, where $v = v_1v_2$. If $\lg(v_1) \geq \lg(v_2)$, then $\lg(u) \leq 0$. This leads to a contradiction. Thus $\lg(v_1) < \lg(v_2)$. As $v_1 <_s x$ and $v_2 <_s x$, there exists $v_3 \in X^+$ such that $v = v_1v_2 = v_1v_3v_1$. The equalities $x = uv_1v_2v_1 = v_2v_1v_2$ imply that $uv_1v_3v_1v_1 = v_3v_1v_1v_3v_1$. Thus we can get $u = v_3$. Hence $(uv_1)^2 \leq_p uv$. This leads to a contradiction. Suppose $\lg(y) \geq \lg(v)$. Then $uv^3 = x^2y$ for some $x, y \in X^+$ implies $x^2 \leq_p uv^2$. This leads to a contradiction. Hence $\lg(y) < \lg(v)$. Let $v = v_1y$ for some $v_1, y \in X^+$. If $\lg(x) = \lg(uv)$, then $\lg(u) = \lg(v_1)$. As $u <_p v$ and $v_1 <_p v$, we get $u = v_1$. Hence $(v_1)^2 <_p uv$. This leads to a contradiction. If $\lg(x) > \lg(uv)$, then there exists $v_2, v_2^* \in X^+$ such that $x = uvv_2$ and $v = v_2v_2^*$. As

$\lg(x) = \lg(uv_2v_2^*v_2) = \lg(v_2^*v_1)$, we get $2\lg(v_2) < \lg(v_1)$. As $v_2 <_p v$ and $v_2 <_s v_1$. Then we can get $v_1 = v_2v_3v_2$, where $v_3 \in X^+$. Thus $v = v_2v_3v_2y$ and $v_2^* = v_3v_2y$. As $\lg(x) = \lg(uv_2v_3v_2yv_2) = \lg(v_3v_2yv_2v_3v_2)$, we get $u = v_3$. Hence $(v_3v_2)^2 <_p uv$. This leads to a contradiction. Therefore $\lg(x) < \lg(uv)$. Then there exists $v_2 \in X^+$ such that $uv = xv_2$ and $\lg(v_2) < \lg(u)$. As $v_2 <_p u$, $u = v_2u_1$ for some $u_1 \in X^+$. If $v = v_1v_2$, then $x = v_2vv_1 = (v_2v_1)^2$. This leads to a contradiction. If $\lg(v) < \lg(v_1v_2)$, then there exists $v_3^* \in X^+$ such that $v = v_3^*v_2$. Thus $\lg(v_3^*) < \lg(v_1)$. As $\lg(x) = \lg(v_2u_1v_3^*) = \lg(v_2v_3^*v_2v_1)$, we can get $\lg(u_1) = \lg(v_2v_1) > \lg(v)$. This leads to a contradiction. Therefore $\lg(v) > \lg(v_1v_2)$. As $v_1v_2 <_s v$, there exists $v_3 \in X^+$ such that $v = v_3v_1v_2$. Consider the following cases:

(1-1) $\lg(v_1) > \lg(v_3)$. Then there exists $v_4 \in X^+$ such that $v_1 = v_3v_4$. If $\lg(v_2) = \lg(v_3)$, then $v_2 = v_3$. As $\lg(x) = \lg(v_2u_1v_3v_1) = \lg(v_2v_3v_1v_2v_1)$, $\lg(u_1) = \lg(v_2v_1) = \lg(v_2v_3v_4)$. Thus $v_3 <_p u_1$. Therefore $(v_2)^2 <_p u$. This leads to a contradiction. If $\lg(v_2) > \lg(v_3)$, then $\lg(u_1) > \lg(v_3v_3v_4)$. Thus $\lg(u) = \lg(v_2u_1) = \lg(v_2v_2v_3v_4) > \lg(v_2v_3v_3v_4) = \lg(v)$. This leads to a contradiction. We must have $\lg(v_2) < \lg(v_3)$. As $\lg(v_1y) = \lg(v) > \lg(v_1) + \lg(v_2)$, there exists $v_5 \in X^+$ such that $y = v_5v_2$. Thus $v = v_1y = v_3v_4v_5v_2$. From $v = v_3v_3v_4v_2$, we get $\lg(v_5) = \lg(v_3)$. Hence there exist $v_6, v_7 \in X^+$ such that $v_5 = v_6v_7$ and $v_3 = v_7v_2$. This implies that $u_1 = v_3v_4v_6 = v_7v_2v_4v_6$. If $\lg(v_4) \geq \lg(v_5)$. Then $v_3^2 <_p u_1$, $v_2v_3^2 <_p u$, i.e., $(v_2v_7)^2 <_p u$. This leads to a contradiction. If $\lg(v_4) < \lg(v_5)$. Since $v_4v_6v_7 = v_3v_4 = v_7v_2v_4$ and $v_2v_7v_2v_4v_6v_7 <_p uv$. Thus $(v_2v_7)^2 <_p uv$. Again, this leads to a contradiction.

(1-2) $\lg(v_1) = \lg(v_3)$. As $v_1 <_p v$ and $v_3 <_p v$, $v_1 = v_3$ and $v = v_3v_1v_2 = v_3v_3v_2$. If $\lg(v_3) = \lg(v_2)$. As $x = v_2u_1v_3v_1 = v_2v_3v_1v_2v_1$, $v_3 = v_2$. This implies that $u_1 = v_3v_1$. Hence $u = v_2u_1 = v_3v_3v_3$. This leads to a contradiction. If $\lg(v_3) > \lg(v_2)$, then there exist $v_4, v_5 \in X^+$ such that $v_3 = v_4v_2 = v_5v_4$ and $u_1 = v_4v_2v_5$. This in conjunction with $u = v_2u_1$ and $v = u_1v_3 = v_3v_3v_2$ implies that $uv = v_2u_1v_3v_3v_2 = v_2v_3v_3v_2v_3v_2 = (v_2v_4)^2v_2v_2v_4v_2v_2$. This leads to a contradiction. If $\lg(v_3) < \lg(v_2)$, then there exists $v_4 \in X^+$ such that $v_2 = v_4v_3$. This implies that $u = v_2u_1 = v_4v_3v_3v_3v_4$ and $v = v_3v_3v_4v_3$. Thus $\lg(u) > \lg(v)$. This leads to a contradiction.

(1-3) $\lg(v_1) < \lg(v_3)$. As $v_1 <_p v$ and $v_3 <_p v$, there exists $v_4 \in X^+$ such that $v_3 = v_1v_4$. If $\lg(v_2) = \lg(v_4)$, then $v_2 = v_4$. Thus $u_1 = v_3 = v_1v_2$ and $v = (v_1v_2)^2$. This implies that $uv = v_2u_1v = (v_2v_1)^3v_2$. This leads to a contradiction. Suppose $\lg(v_2) > \lg(v_4)$. If $\lg(v_2) = \lg(v_1v_4)$, then $v_2 = v_1v_4$. Thus $u_1 = v_3v_1 = v_1v_4v_1$. This implies that $u = v_2u_1 = (v_1v_4)^2v_1$. This leads to a contradiction. If $\lg(v_2) > \lg(v_1v_4)$, then $\lg(u_1) > \lg(v_3v_1)$. Thus $\lg(u) = \lg(v_2u_1) > \lg(v_2v_3v_1) = \lg(v)$. This leads to a contradiction. If $\lg(v_2) < \lg(v_1v_4)$, then $\lg(u_1) = \lg(v_1v_2) > \lg(v_1v_4)$ implies that there exist $v_5, v_6, v_7 \in X^+$ such that $u_1 = v_1v_4v_5$, $v_1 = v_5v_6 = v_6v_7$ and $v_2 = v_7v_4$. Thus $u = v_2u_1 = v_2v_1v_4v_5 = v_7v_4v_6v_7v_4v_5$ and $v = v_6v_7v_4v_5v_6v_7v_4$. This implies that $(v_7v_4v_6)^2 <_p uv$. This leads to a contradiction. Therefore $\lg(v_2) < \lg(v_4)$. Then there exists $v_5 \in X^+$ such that $v_4 = v_5v_2$. Hence $v_3 = v_1v_5v_2$. As $v = u_1v_3 = v_3v_1v_2$, $u_1v_1v_5v_2 = v_1v_5v_2v_1v_2$. Thus $v_1 <_p u_1$. Then there exists $v_6 \in X^+$ such that $u_1 = v_1v_6$. Therefore $v_1v_6v_1v_5v_2 = v_1v_5v_2v_1v_2$. So we can get $v_6v_1v_5 = v_5v_2v_1$. Consider the following subcases:

(1-3-1) $\lg(v_6) = \lg(v_5)$. Then $v_6 = v_5$ and $v_1v_6 = v_1v_5 = v_2v_1$. This implies that $u = v_2u_1 = v_2v_1v_6 = v_2v_2v_1$. This leads to a contradiction.

(1-3-2) $\lg(v_1) = \lg(v_5)$. Then $v_1 = v_5$. If $\lg(v_1) = \lg(v_2)$, then $v_1 = v_2 = v_5 = v_6$. Thus $u = v_2u_1 = v_2v_1v_6 = (v_1)^3$. This leads to a contradiction. If $\lg(v_1) < \lg(v_2)$, then there exists $v_7 \in X^+$ such that $v_2 = v_7v_1$ and $v_6 = v_5v_7 = v_1v_7$. So $u_1 = v_1v_6 = v_1v_1v_7$. Thus $u = v_2u_1 = v_7v_1v_1v_1v_7$ and $v = u_1v_1v_5v_2 = (v_1v_1v_7)^2v_1$. The assertion with statement (1) holds, where $x_1 = v_7, x_2 = x_3 = v_1$ and $k = 2$. If $\lg(v_1) > \lg(v_2)$, then there exist $v_7 \in X^+$ such that $v_1 = v_5 = v_7v_2 = v_6v_7$. Thus there exist $w_1, w_2 \in X^+$, $n \geq 0$ such that $v_6 = w_1w_2$, $v_7 = (w_1w_2)^nw_1$ and $v_2 = w_2w_1$. Hence $u = v_2u_1 = v_2v_1v_6 = w_2w_1(w_1w_2)^{n+1}w_1w_1w_2$ and $v = v_1v_5v_2v_1v_2 = ((w_1w_2)^{n+1}w_1w_1w_2)^2(w_1w_2)^{n+1}w_1$. The assertion with statement (3) holds, where x_4 is an empty word, $i = 1$ and $k = 2$.

(1-3-3) $\lg(v_1) = \lg(v_2)$. Then $\lg(v_1) = \lg(v_2) = \lg(v_6)$. If $\lg(v_5) = \lg(v_6)$, by an analogous argument as case (1-3-1), then this leads to a contradiction. If $\lg(v_5) > \lg(v_6)$, then there exist $v_7, v_8, v_9 \in X^+$ such that $v_5 = v_6v_7$, $v_1 = v_7v_8$ and $v_2 = v_8v_9$. This implies that $v_5 = v_6v_7 = v_9v_1 = v_9v_7v_8$. Case 1: $\lg(v_7) = \lg(v_8)$. Then $v_7 = v_8$ and $v_6 = v_9v_7$. Thus $v_2 = v_7v_9$ and $u_1 = v_1v_6 = v_7v_7v_9v_7$. This implies that $u = v_2u_1 = (v_7v_9v_7)^2$. This leads to a contradiction. Case 2: $\lg(v_7) < \lg(v_8)$. Then there exists $v_{10} \in X^+$ such that $v_8 = v_{10}v_7$. This implies that $v_5 = v_6v_7 = v_9v_7v_{10}v_7$. Thus $v_6 = v_9v_7v_{10}$ and $u_1 = v_1v_6 = v_7v_{10}v_7v_9v_7v_{10}$. This implies that $u = v_2u_1 = v_8v_9u_1 = (v_{10}v_7v_9v_7)^2v_{10}$. This leads to a contradiction. Case 3: $\lg(v_7) > \lg(v_8)$. Then there exist $v_{10}, v_{11} \in X^+$ such that $v_7 = v_{10}v_{11} = v_{11}v_8$. So there exist $w_1, w_2 \in X^+$, $n \geq 0$ such that $v_{10} = w_1w_2$, $v_{11} = (w_1w_2)^nw_1$ and $v_8 = w_2w_1$. Thus $v_7 = (w_1w_2)^{n+1}w_1$, $v_6 = v_9w_1w_2$ and $u_1 = (w_1w_2)^{n+2}w_1v_9w_1w_2$. Hence $u = v_2u_1 = v_2v_1v_7 = w_2w_1v_9(w_1w_2)^{n+2}w_1v_9w_1w_2$ and $v = v_1v_5v_2v_1v_2 = ((w_1w_2)^{n+2}w_1v_9w_1w_2)^2(w_1w_2)^{n+2}w_1v_9$. The assertion with statement (3) holds, where $x_4 = v_9$, $i = 2$ and $k = 2$. If $\lg(v_5) < \lg(v_6)$, then there exist $v_7, v_8, v_9 \in X^+$ such that $v_6 = v_5v_7$, $v_2 = v_7v_8$ and $v_1 = v_8v_9$. This implies that $v_1 = v_8v_9 = v_9v_5$. So there exist $w_1, w_2 \in X^+$, $n \geq 0$ such that $v_8 = w_1w_2$, $v_9 = (w_1w_2)^nw_1$ and $v_5 = w_2w_1$. Thus $v_1 = (w_1w_2)^{n+1}w_1$ and $u_1 = v_1v_6 = v_1v_5v_7 = (w_1w_2)^{n+1}w_1w_2w_1v_7$. Hence $u = v_2u_1 = v_7w_1w_2(w_1w_2)^{n+2}w_1v_7$ and $v = v_1v_5v_2v_1v_2 = ((w_1w_2)^{n+2}w_1v_7)^2w_1w_2$. The assertion with statement (4) holds, where $x_3 = v_7$, $i = 2$ and $k = 2$.

(1-3-4) $\lg(v_1) < \lg(v_2) < \lg(v_5)$. As $v_6v_1v_5 = v_5v_2v_1$ and $\lg(v_2) = \lg(v_6) < \lg(v_5)$, there exists $v_7 \in X^+$ such that $v_5 = v_6v_7$. Thus $v_1v_6v_7 = v_7v_2v_1$ and $\lg(v_6) = \lg(v_2)$. Case 1: $\lg(v_1) = \lg(v_7)$. Then $v_1 = v_7$ and $v_2 = v_6$. Thus $u = v_2u_1 = v_2v_1v_6 = v_2v_1v_2$, $v = v_1v_5v_2v_1v_2 = (v_1v_2)^3$. Hence $uv = (v_2v_1)^4v_2$. This leads to a contradiction. Case 2: $\lg(v_1) > \lg(v_7)$. Then there exist $v_8, v_9, v_{10} \in X^+$ such that $v_1 = v_7v_8 = v_{10}v_7$, $v_6 = v_9v_{10}$ and $v_2 = v_8v_9$. Thus there exist $w_1, w_2 \in X^+$, $n \geq 0$ such that $v_{10} = w_1w_2$, $v_7 = (w_1w_2)^nw_1$ and $v_8 = w_2w_1$. This implies that $u_1 = v_1v_6 = v_{10}v_7v_9v_{10} = (w_1w_2)^{n+1}w_1v_9w_1w_2$. Hence $u = v_2u_1 = v_8v_9u_1 = w_2w_1v_9(w_1w_2)^{n+1}w_1v_9w_1w_2$ and $v = v_1v_5v_2v_1v_2 = ((w_1w_2)^{n+1}w_1v_9w_1w_2)^2(w_1w_2)^{n+1}w_1v_9$. The assertion with statement (3) holds, where $x_4 = v_9$, $i = 1$ and $k = 2$. Case 3: $\lg(v_1) < \lg(v_7)$. Then there exist $v_8, v_9, v_{10}, \in X^+$ such that $v_7 = v_1v_8 = v_9v_1$ and $v_6 = v_{10}v_9$. Thus there exist $w_1, w_2 \in X^+$ such that $v_9 = w_1w_2$, $v_1 = (w_1w_2)^nw_1$ and $v_8 = w_2w_1$.

Hence $u = v_2v_1v_6 = v_{10}v_9v_1v_8v_{10} = v_{10}w_1w_2(w_1w_2)^{n+1}w_1v_{10}$ and $v = v_1v_5v_2v_1v_2 = v_1v_6v_7v_2v_1v_2 = ((w_1w_2)^{n+1}w_1v_{10})^3w_1w_2$. The assertion with statement (4) holds, where $x_3 = v_{10}$, $i = 1$ and $k = 3$.

(1-3-5) $\lg(v_1) < \lg(v_5) < \lg(v_2)$. Then there exist $v_7, v_8 \in X^+$ such that $v_5 = v_7v_1$, $v_2 = v_8v_1v_7$ and $v_6 = v_5v_8$. Hence $u = v_8v_1v_7v_1v_7v_1v_8$ and $v = (v_1v_7v_1v_8)^2v_1v_7$. The assertion with statement (1) holds, where $x_1 = v_8$, $x_2 = v_1v_7$, $x_3 = v_1$ and $k = 2$.

(1-3-6) $\lg(v_2) < \lg(v_1) < \lg(v_5)$. If $\lg(v_5) = \lg(v_6v_1)$, then $v_5 = v_6v_1 = v_2v_1$. This implies that $v_6 = v_2$. Thus $u = v_2v_1v_6 = v_2v_1v_2$. Hence $(v_2v_1)^2 <_{\mathrm{p}} uv$. This leads to a contradiction. If $\lg(v_5) < \lg(v_6v_1)$, then there exist $v_7, v_8 \in X^+$ such that $v_1 = v_7v_8$, $v_5 = v_6v_7 = v_9v_1 = v_9v_7v_8$ and $v_2 = v_8v_9$. Thus $u = v_2v_1v_6 = v_8v_9v_7v_8v_6$. Therefore $uv_1 = v_8v_9v_7v_8v_6v_7v_8 = (v_8v_9v_7)^2v_8v_8$. Since $uv_1 <_{\mathrm{p}} uv$. Hence $(v_8v_9v_7)^2 <_{\mathrm{p}} uv$. This leads to a contradiction. Therefore $\lg(v_5) > \lg(v_6v_1)$. Then there exists $v_7 \in X^+$ such that $v_5 = v_6v_1v_7 = v_7v_2v_1$. If $\lg(v_6) = \lg(v_7)$, then $v_6 = v_7$ and $v_2v_1 = v_1v_7$. Thus there exist $w_1, w_2 \in X^+$ such that $v_2 = w_1w_2$, $v_1 = (w_1w_2)^nw_1$ and $v_7 = w_2w_1$. Hence $(w_1w_2)^2 <_{\mathrm{p}} u$. This leads to a contradiction. If $\lg(v_6) > \lg(v_7)$, then there exist $v_8, v_9, v_{10} \in X^+$ such that $v_6 = v_7v_8, v_2 = v_8v_9$ and $v_1 = v_9v_{10} = v_{10}v_7$. Thus there exist $w_1, w_2 \in X^+$ such that $v_9 = w_1w_2$, $v_{10} = (w_1w_2)^nw_1$ and $v_7 = w_2w_1$. Hence $u = v_2v_1v_6 = v_8w_1w_2(w_1w_2)^{n+2}w_1v_8$ and $v = v_1v_5v_2v_1v_2 = ((w_1w_2)^{n+2}w_1v_8)^3w_1w_2$. The assertion with statement (4) holds, where $x_3 = v_8$, $i = 2$ and $k = 3$. If $\lg(v_6) < \lg(v_7)$, then there exist $v_8 \in X^+$ such that $v_7 = v_6v_8$ and $v_1v_7 = v_8v_2v_1$. Thus $v_1v_6v_8 = v_8v_2v_1$. If $\lg(v_1) = \lg(v_8)$, then $v_1 = v_8$ and $v_2 = v_6$. Thus $u = v_2v_1v_6 = v_2v_1v_2$. Since $uv_1 <_{\mathrm{p}} uv$, $(v_2v_1)^2 <_{\mathrm{p}} uv$. This leads to a contradiction. If $\lg(v_1) < \lg(v_8)$, then there exist $v_9, v_{10}, v_{11} \in X^+$ such that $v_8 = v_1v_9 = v_{10}v_1$, $v_6 = v_9v_{11}$ and $v_2 = v_{11}v_{10}$. Thus there exist $w_1, w_2 \in X^+$ such that $v_{10} = w_1w_2$, $v_1 = (w_1w_2)^nw_1$ and $v_9 = w_2w_1$. Hence $u = v_2v_1v_6 = v_{11}w_1w_2(w_1w_2)^{n+1}w_1v_{11}$ and $v = v_1v_5v_2v_1v_2 = ((w_1w_2)^{n+1}w_1v_{11})^4w_1w_2$. The assertion with statement (4) holds, where $x_3 = v_{11}$, $i = 1$ and $k = 4$. If $\lg(v_1) > \lg(v_8)$, then there exist $v_9, v_{10}, v_{11} \in X^+$ such that $v_1 = v_8v_9$, $v_2 = v_9v_{10}$, $v_6 = v_{10}v_{11}$. Therefore $v_1 = v_{11}v_8 = v_8v_9$. Thus there exist $w_1, w_2 \in X^+$ such that $v_{11} = w_1w_2$, $v_8 = (w_1w_2)^nw_1$ and $v_9 = w_2w_1$. Hence $u = v_2v_1v_6 = w_2w_1v_{10}(w_1w_2)^{n+1}w_1v_{10}w_1w_2$ and $v = v_1v_5v_2v_1v_2 = ((w_1w_2)^{n+1}w_1v_{10}w_1w_2)^3(w_1w_2)^{n+1}w_1v_{10}$. The assertion with statement (3) holds, where $x_4 = v_{10}$, $i = 1$ and $k = 3$.

(1-3-7) $\lg(v_2) < \lg(v_5) < \lg(v_1)$. As $v_6v_1v_5 = v_5v_2v_1$, there exist $v_7, v_8 \in X^+$ such that $v_5 = v_6v_7$, $v_1 = v_7v_2v_8 = v_8v_5 = v_8v_6v_7$. Thus we can get $v_7v_2v_8v_6 = v_8v_6v_7v_6$. Hence $u = v_2v_1v_6 = v_2v_7v_2v_8v_6$ and $v = v_1v_5v_2v_1v_2 = (v_7v_2v_8v_6)^2v_7v_2$. The assertion with statement (2) holds, where $x_1 = v_2, x_2 = v_7$ and $x_3 = v_8v_6$.

(1-3-8) $\lg(v_5) < \lg(v_2) < \lg(v_1)$ and $\lg(v_5) < \lg(v_1) < \lg(v_2)$. As $v_6v_1v_5 = v_5v_2v_1$, there exist $v_7, v_8, v_9 \in X^+$ such that $v_6 = v_5v_7$ and $v_2 = v_7v_8$ and $v_1 = v_8v_9 = v_9v_5$. Therefore there exist $w_1, w_2 \in X^+$, $n \geq 0$ such that $v_8 = w_1w_2$, $v_9 = (w_1w_2)^nw_1$ and $v_5 = w_2w_1$. Hence $u = v_2v_1v_6 = v_7v_8v_1v_5v_7 = v_7w_1w_2(w_1w_2)^{n+2}w_1v_7$ and $v = v_1v_5v_2v_1v_2 = ((w_1w_2)^{n+2}w_1v_7)^2w_1w_2$. The assertion with statement (4) holds, where $x_3 = v_7$, $i = 2$ and $k = 2$.

($\Leftarrow$) Immediate. ∎

Next, we show that uv^n are p-primitive words for all $n \geq 4$ for any two distinct words

u, v with $u, uv, uv^2, uv^3 \in P_1(X)$ and $\lg(u) < \lg(v)$.

Proposition 2.4 Let u, v be two distinct words such that $u, uv, uv^2, uv^3 \in P_1(X)$ and $\lg(u) < \lg(v)$. Then $uv^n \in P_1(X)$ for all $n \geq 4$.

Proof. As uv^n is not a p-primitive word for all $n \geq 4$, $uv^n = x^2y$ for some $x \in X^+$ and $y \in X^*$. As $uv^n = x^2$. Then either $x = uv^iv_1 = v_2v^i$ or $x = uv^iv_1 = v_2v^{i+1}$ may occur, where $v = v_1v_2$, $v_1, v_2 \in X^+$ and $i > 0$. Suppose $x = uv^iv_1 = v_2v^i$. Then $v_2 = uv_1$. Thus $uv = uv_1v_2 = (uv_1)^2$. This leads to a contradiction. Suppose $x = uv^iv_1 = v_2v^{i+1}$, where $2i + 2 = n$. Therefore $\lg(u) = 2\lg(v_2)$. As $v_2 <_p x$ and $u <_p x$, there exists $u_1 \in X^+$ such that $u = v_2u_1$ and $\lg(v_2) = \lg(u_1)$. Since $\lg(u) < \lg(v)$, i.e., $\lg(v_2u_1) < \lg(v_1v_2)$, we get $\lg(u_1) < \lg(v_1)$. As $u_1 <_p v$, there exists $v_3 \in X^+$ such that $v_1 = u_1v_3$. This together with $u_1v_3 <_s x$, $v_3v_2 <_s x$ and $\lg(u_1v_3) = \lg(v_3v_2)$ yield $u_1v_3 = v_3v_2$. Hence $u = v_2u_1$, $v = u_1v_3v_2$ and $u_1v_3 = v_3v_2$. This implies that $uv^2 = v_2u_1u_1v_3v_2u_1v_3v_2 = v_2u_1v_3v_2v_2u_1v_3v_2 = (v_2u_1v_3v_2)^2 \notin P_1(x)$. This leads to a contradiction. Let $uv^n = x^2y$ for some $x, y \in X^+$. If $u <_p v$, then $uv \notin P_1(x)$. This leads to a contradiction. If $u <_s v$, then $uv^2 \notin P_1(x)$. This leads to a contradiction. Therefore $u \not<_p v$ and $u \not<_s v$. Case 1: There exist $v_1, v_2 \in X^+$ such that $v = v_1uv_2$. If $\lg(v_1) \leq \lg(v_2)$, then $v_1 <_p v$ and $v_2 <_p v$ imply that there exists $v_3 \in X^*$ such that $v_2 = v_1v_3$. Thus $v = v_1uv_1v_3$. Therefore $uv \notin P_1(x)$. This leads to a contradiction. If $\lg(v_1) > \lg(v_2)$, then there exists $v_3 \in X^+$ such that $v_1 = v_2v_3$. Thus $v = v_1uv_2 = v_2v_3uv_2 = v_3uv_2v_2$. This implies that $v_2v_3u = v_3uv_2$. Therefore there exist $w \in X^+$ and $m, q \geq 1$ such that $v_2 = w^m$ and $v_3u = w^q$. There also exist $w_1, w_2 \in X^*$ and $q_1, q_2 \geq 0$ such that $v_3 = w^{q_1}w_1$ and $u = w_2w^{q_2}$, where $w = w_1w_2$ and $q = q_1 + q_2 + 1$. Thus $v = w^{2m+q}$. If $q_2 = 0$, then $u = w_2$. Therefore $(w_2w_1)^2 <_p uv$. This leads to a contradiction. If $q_2 = 1$, then $u = w_2w_1w_2$. As $w_1 <_p v$, $(w_2w_1)^2 <_p uv$. This leads to a contradiction. If $q_2 \geq 2$, then $u = w_2w^2w^{q_2-2}$. Therefore $(w_2w_1)^2 <_p u$. This leads to a contradiction. Case 2: If there exist $u_1, u_2 \in X^*$ such that $u = u_1u_2$, $u_1 <_s v$ and $u_2 <_p v$, then there exists $v_1 \in X^+$ such that $v = u_2v_1u_1$. We get $u_2v_1u_1 = v_1u_1u_2$. Thus $uv^2 = u_1u_2u_2v_1u_1u_2v_1u_1 = u_1u_2v_1u_1u_2v_1u_1u_2 = (u_1u_2v_1)^2u_1u_2$. This leads to a contradiction. ∎

3. Conclusion

Proposition 2.4 tells us that when we want to check whether or not uv^+ is a p-primitive word for the case $\lg(u) < \lg(v)$, we just check the characters of u and v whether they are in the statements of Proposition 2.1 to Proposition 2.3 or not. We conjecture that the cases for $\lg(u) = \lg(v)$ and $\lg(u) > \lg(v)$ have same results which are left for our further research.

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