Solving Boundary Value Problems Involving Heat Distributions in Lumped Rods

差分方程式解熱傳導邊界值問題

樂文全

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Abstract

The heat transfer problems in thermodynamics are usually modeled by partial differential equations. In recent years, the theory of difference equations has found its applications in many aspects [1,2,3]. In this paper, we derived solutions for the difference equations of one-dimensional heat transfer boundary value problems. Symbolic operator theory and orthogonality formulas are employed.

摘要

熱力學中有關熱傳導方程式的邊界值問題，一般都利用微分方程式求解。近年來，以差分方程式為基礎之運算符號法發展迅速，並且應用在許多方面。本文推導出某些有關對稱矩陣的正交性公式，並應用於以差分方程式之運算符號法解一維熱傳導方程式的邊界值問題，獲得滿意之結果。

Key words: Sequence, convolutional product, symmetric matrix, orthogonality, difference equation, heat transfer equation.

1. Introduction

An ordered set of numbers (real or complex) form a sequence, which may symbolically be denoted by

$$f = \left\{ f_0, f_1, f_2, \ldots, f_k, \ldots \right\} = \left\{ f_k \right\}_{k=0}^{\infty}.$$

Two sequences may be added in the usual manner

$$(f+g)_k = f_k + g_k, \ k = 0, 1, 2, \ldots$$

and multiplied under "convolutional" product

$$(f * g)_k = \sum_{i=0}^{k} f_{k-i} \cdot g_i, \ k = 0, 1, 2, \ldots \quad ,$$

where $(f+g)_k$ and $(f*g)_k$ denotes the k-th term in the sequence $f+g$ or $f*g$ respectively. A sequence with the first term ($k=0$) equals to a and all the remaining terms equal to zero is denoted by $\bar{a}$, that is

$$\bar{a}=\{a,0,0,\ \ldots,\ 0,\ \ldots\}=a\{1,0,0,\ \ldots,0,\ \ldots\}=a\,\bar{1},$$

where $\bar{1}=\{1,0,0,0,\ldots\}$ is often written as 1 for simplicity.

An important sequence $\hbar$ is defined by

$$\hbar=\{0,1,0,0,\ldots,0,\ldots\}.$$

By applying the convolutional product defined above, we have

$$\hbar^2=\{0,0,1,0,0,\ldots\},$$
$$\hbar^3=\{0,0,0,1,0,\ldots\},$$

and in general

$$\hbar^m=\{d_k^m\}_{k=0}^{\infty},$$

where

$$d_k^m=\begin{cases}1 & k=m,\\ 0 & k\neq m,\end{cases}$$

and $\hbar^0$ is defined by $\{1,0,0,0,\ldots\}=1$. In this sense, $\hbar^m$ is often called the Dirac sequence. The convolutional product of $\hbar^m$ with a sequence f is

$$\begin{aligned}\hbar^m*f&=\{0,0,\ldots,1,0,\ldots\}*\{f_0,f_1,f_2,\ldots,f_k,\ldots\}\\&=\{0,0,\ldots,f_0,f_1,f_2,\ldots\},\end{aligned}$$

which may be obtained by shifting every term in f by m positions to the right and filling zeros in the first m empty positions. The product sign $*$ is usually omitted, that is, $\hbar^m*f$ is usually designated as $\hbar^m f$ for simplicity. $\hbar^m$ is usually treated as an operator.

Another important operator known as the shifting operator is defined by

$$E\{f_k\}_{k=0}^{\infty}=\{f_{k+1}\}_{k=0}^{\infty}=\{f_1,f_2,f_3,\ldots,f_k,\ldots\}$$

or simply

$$E\,f=\{f_{k+1}\}_{k=0}^{\infty}.$$

Applying this operator m times to a sequence f yields

$$E^m\{f_k\}_{k=0}^{\infty}=\{f_{k+m}\}_{k=0}^{\infty}=\{f_m,f_{m+1},f_{m+2},\ldots,f_{m+k},\ldots\},\quad m\geq 0,$$

which is obtained by shifting every term in f by m positions to the left. In this sense, the sequence $E^{-m}f$ is defined by shifting every term in f to the right and m zeros are added before f_0. That is,

$$E^{-m}f=\{0,0,\ldots,f_0,f_1,f_2,\ldots\},$$

which is exactly the same as $\hbar^m f$. Hence $E^{-m}=\hbar^m$. Note that $E^m\hbar^m f=f$ but $\hbar^m E^m f\neq f$. To see this, note that

$$\hbar\,E\,f=\{0,f_1,f_2,f_3,f_4,\ldots\}=f-f_0\hbar^0,$$

$$\hbar^2 E^2 f = \{0, 0, f_1, f_2, f_3, f_4, \dots\} = f - f_0\hbar^0 - f_1\hbar^1,$$
$$\hbar^3 E^3 f = \{0, 0, 0, f_3, f_4, \dots\} = f - f_0\hbar^0 - f_1\hbar^1 - f_2\hbar^2,$$

and in general

$$\hbar^m E^m f = \hbar^m \{f_m, f_{m+1}, \dots, f_{m+k}, \dots\}_{k=0}^{\infty} = \{0, 0, \dots, f_m, f_{m+1}, \dots, f_{m+k}, \dots\}, \quad m \ge 0 .$$

So we have

$$\hbar^m E^m f = f - f_0\hbar^0 - f_1\hbar^1 - f_2\hbar^2 - \dots - f_{m-1}\hbar^{m-1} = f - \sum_{i=0}^{m-1} f_i \hbar^i . \qquad \text{(A)}$$

The Dirac sequence operator $\hbar^m$ may also be treated as an ordinary variable in a formula. For instance,

$$\frac{1}{1-a\hbar} = 1 + a\hbar + (a\hbar)^2 + \dots$$
$$= \sum_{k=0}^{\infty} (a\hbar)^k = \{1, a, a^2, \dots\} = \{a^k\}_{k=0}^{\infty}, \quad a \ne 0,$$

and

$$\frac{1}{a-\hbar} = \frac{1}{a} \cdot \frac{1}{1-\frac{1}{a}\hbar} = \frac{1}{a}\left(1 + \frac{1}{a}\hbar + \left(\frac{1}{a}\hbar\right)^2 + \dots\right)$$

$$= \frac{1}{a}\{a^{-k}\}_{k=0}^{\infty} = \{a^{-k-1}\}_{k=0}^{\infty}, \quad a \ne 0 .$$

The "derivative" of a sequence is defined by

$$D\{f_k\}_{k=0}^{\infty} = \{(k+1) f_{k+1}\}_{k=0}^{\infty} .$$

In particular,

$$D\hbar^m = m\hbar^{m-1}$$

and

$$D\left(\frac{1}{1-\hbar}\right) = \sum_{k=0}^{\infty} (k+1)\hbar^k = \{1, 2, 3, \dots, k+1, \dots\}. \qquad \text{(B)}$$

The comp lete theory that explains our approach can be found in Cheng [3].

2. The one dimensional heat transfer equation

Consider the heat distribution problem for a finite "lumped rod”. Suppose the lumps on the rod can be labeled by a consecutive set of integers. Let $U_k^{(t)}$ be the temperature at the integral position k and integral time t of the rod. At time t, if the temperature $U_{k-1}^{(t)}$ is higher than $U_k^{(t)}$, heat will flow from the point $k-1$ to k. The amount of increase in temperature is $U_k^{(t+1)} - U_k^{(t)}$, and it is reasonable to postulate that the increase is proportional to the difference $U_{k-1}^{(t)} - U_k^{(t)}$, say, $d(U_{k-1}^{(t)} - U_k^{(t)})$ where d is a positive diffusive constant. Similarly, heat will flow from the point $k+1$ to k. Thus it is reasonable that the total effect is

$$U_k^{(t+1)} - U_k^{(t)} = d\left(U_{k-1}^{(t)} - U_k^{(t)}\right) + d\left(U_{k+1}^{(t)} - U_k^{(t)}\right)$$

or

$$U_k^{(t+1)} - U_k^{(t)} = d\left(U_{k-1}^{(t)} - 2U_k^{(t)} + U_{k+1}^{(t)}\right). \tag{1}$$

In steady state, $U_k^{(t+1)} - U_k^{(t)} = 0$. Hence

$$U_{k-1}^{(t)} - 2U_k^{(t)} + U_{k+1}^{(t)} = 0 . \tag{2}$$

This equation can be solved by the method of "separation of variables" as described in [3]. Let $U_k^{(t)} = a_t b_k$, substituting this into (2), we have

$$b_{k-1} - 2b_k + b_{k+1} = 0. \tag{3}$$

By applying the shifting operator E, equation (3) may be written as

$$E^2 b - 2Eb + b = 0, \tag{4}$$

where

$$b = \{b_k\}_{k=0}^{\infty}.$$

Multiplying each term of (4) by $\hbar^2$, and making use of the formula (A), we have

$$(b - b_0 - b_1\hbar) - 2\hbar(b - b_0) + \hbar^2 b = 0 .$$

Solving for b , we get

$$b = \frac{c}{\hbar^2 - 2\hbar + 1} = \frac{c}{(\hbar - 1)^2} = D\left(\frac{c}{1-\hbar}\right)$$

where

$$c = b_0 + (b_1 - 2b_0)\hbar .$$

By evaluating the convolutional product of c with $D\left(\frac{1}{1-\hbar}\right)$, which is given by (B), we obtain

$$b = \{b_0, (b_1 - 2b_0), 0, 0, \ldots\} * \{1, 2, \ldots, (k+1), \ldots\}$$

$$= \{b_0(k+1) + (b_1 - 2b_0)\, k\}_{k=0}^{\infty} .$$

And the solution of (2) takes the form

$$U_k^{(t)} = a_t(b_0 + (b_1 - b_0)k),$$

where b_0 and b_1 are two undetermined coefficients that can be obtained by subsidiary conditions.

Suppose the boundary conditions are given by

$$U_0^{(t)} = A_0, \quad U_L^{(t)} = B_0,$$

then

$$U_0^{(t)} = a_t b_0 = A_0$$

and

$$U_L^{(t)} = a_t(b_0 + (b_1 - b_0)L) = B_0$$

or

$$a_t b_1 = \frac{1}{L}(B_0 + (L-1)A_0) = \frac{B_0 - A_0}{L} + A_0 .$$

Hence the solution of (2) for the steady state becomes

$$U_k^{(t)} = A_0 + \frac{B_0 - A_0}{L} k = A_0 + C_0 k\,, \quad k = 0, 1, 2, \dots, L\,; \quad t = 0, 1, 2, \dots \quad , \tag{6}$$

where $C_0 = \dfrac{B_0 - A_0}{L}$.

In general $U_k^{(t+1)} - U_k^{(t)} \neq 0$. Equation (1) may be written as

$$U_k^{(t+1)} = dU_{k-1}^{(t)} + (1-2d)U_k^{(t)} + dU_{k+1}^{(t)}. \tag{7}$$

Let $U_k^{(t)} = a_t b_k$ as before, we have

$$a_{t+1} b_k = d a_t b_{k-1} + (1-2d) a_t b_k + d a_t b_{k+1}.$$

Dividing each term by a_t and letting

$$\frac{a_{t+1}}{a_t} = \frac{d\, b_{k-1} + (1-2d) b_k + d\, b_{k+1}}{b_k} = l\ . \tag{8}$$

There are three cases to consider.

If $|l| > 1$, then

$$a_{t+1} = l a_t = l^2 a_{t-1} = \dots\dots = l^{t+1} a_0.$$

We see that $a_t \to \infty$, as $t \to \infty$. Hence (7) cannot have a solution with $|l| > 1$.

If $|l| = 1$, then

$$a_{t+1} = a_t = \dots\dots = a_0 = \text{constant}.$$

And (8) is reduced to (3) since $d \neq 0$. Hence in this case (1) has a 'steady state' solution of the form

$$U_k^{(t)} = A + Ck\,, \quad k = 0, 1, 2, \dots, L\,; \quad t = 0, 1, 2, \dots \quad .$$

If $|l| < 1$, then $a_t = l^t a_0$ and $b_{k-1} + g\, b_k + b_{k+1} = 0$,

where

$$g = \frac{1 - 2d - l}{d}\,.$$

By the same procedure as we have used in deriving from (3) to (6), we see that

$$b = \frac{c}{\hbar^2 + g\,\hbar + 1} = \frac{c}{a-b}\left(\frac{1}{\hbar - a} - \frac{1}{\hbar - b}\right), \quad a \neq b \tag{9}$$

where

$$c = b_0 + (b_1 + b_0 g)\hbar$$

and

$$a = -\frac{g}{2} + \sqrt{\left(\frac{g}{2}\right)^2 - 1}\,, \quad b = -\frac{g}{2} - \sqrt{\left(\frac{g}{2}\right)^2 - 1}\,, \quad ab = 1\,. \tag{10}$$

Since $\dfrac{1}{\hbar - a} = \left\{-a^{-k-1}\right\}$, $\dfrac{1}{\hbar - b} = \left\{-b^{-k-1}\right\}$ and $b = a^{-1}$, we have

$$b = \frac{c}{a-b}\left\{-a^{-k-1} + b^{-k-1}\right\} = \frac{c}{a-b}\left\{a^{k+1} - b^{k+1}\right\}.$$

By evaluating the convolutional product of $\dfrac{c}{a-b}$ with $\left\{a^{k+1}-b^{k+1}\right\}$, we obtain

$$b=\frac{1}{a-b}\left\{b_0, b_1+b_0 g, 0, 0, \ldots\right\} *\left\{a-b, a^2-b^2, \ldots, a^{k+1}-b^{k+1}, \ldots\right\}$$

$$=\frac{1}{a-b}\left\{b_0\left(a^{k+1}-b^{k+1}\right)+\left(b_1+b_0 g\right)\left(a^k-b^k\right)\right\}_{k=0}^{\infty}. \tag{11}$$

Hence (1) has a 'transient' solution

$$U_k^{(t)}=\frac{l^t a_0}{a-b}\left(b_0\left(a^{k+1}-b^{k+1}\right)+\left(b_1+b_0 g\right)\left(a^k-b^k\right)\right).$$

By superposition, the general solution of (1) will be

$$U_k^{(t)}=A+Ck+\frac{l^t a_0}{a-b}\left(b_0\left(a^{k+1}-b^{k+1}\right)+\left(b_1+b_0 g\right)\left(a^k-b^k\right)\right), \tag{12}$$

where A, C and b_0, b_1 are four coefficients to be determined by initial and boundary conditions.

Suppose the boundary conditions for (1) are given by

$$U_0^{(t)}=A_1,\ U_L^{(t)}=B_1,\quad t=0,1,2,\ldots\ .$$

Then, as $t \to \infty$, $l^t=0$ so that $U_0^{(\infty)}=A=A_1$, $U_L^{(\infty)}=A+CL=B_1$, hence

$$C=\frac{B_1-A_1}{L}$$

and

$$U_k^{(t)}=A_1+\frac{B_1-A_1}{L}k+\frac{l^t a_0}{a-b}\left(b_0\left(a^{k+1}-b^{k+1}\right)+\left(b_1+b_0 g\right)\left(a^k-b^k\right)\right).$$

For $t \geq 0$, the boundary condition $U_0^{(t)}=A_1$ demands

$$U_0^{(t)}=A_1+l^t a_0 b_0=A_1.$$

This leads to $l^t a_0 b_0=0$ or $b_0=0$. And (12) is reduced to

$$U_k^{(t)}=A_1+\frac{B_1-A_1}{L}k+\frac{l^t a_0}{a-b}\cdot b_1\left(a^k-b^k\right). \tag{13}$$

The other boundary condition $U_L^{(t)}=B_1$ demands

$$U_L^{(t)}=A_1+\frac{B_1-A_1}{L}\cdot L+\frac{l^t a_0}{a-b}\left[b_1\left(a^L-b^L\right)\right]=B_1.$$

Hence $\dfrac{l^t a_0}{a-b} b_1\left(a^L-b^L\right)=0$. Since $l^t a_0 b_1 \neq 0$, else $U_k^{(t)} \equiv 0$, so we have

$$\frac{a^L-b^L}{a-b}=0.$$

Since $b=a^{-1}$, this may be written as

$$\frac{a^{2L}-1}{a^2-1}=0.$$

The left hand side is a polynomial of a with degree $2L-2$, which has $2L-2$ roots if it has solutions.

Now suppose $\left|\frac{g}{2}\right| > 1$, then $a = -\frac{g}{2} + \sqrt{\left(\frac{g}{2}\right)^2 - 1} < 0$ is a real number, and

$$\frac{a^L - b^L}{a-b} = a^{L-1} + a^{L-2}b + a^{L-3}b^2 + \ldots + ab^{L-2} + b^{L-1}$$
$$= a^{L-1} + a^{L-3} + a^{L-5} + \ldots + a^{-L+3} + a^{-L+1}$$
$$= a^{-L+1}\left(a^{2L-2} + a^{2L-4} + \ldots + a^4 + a^2 + 1\right) \neq 0$$

unless $a = 0$. Consequently, for the equation to have a solution, $\left|\frac{g}{2}\right|$ cannot be greater than 1. Also in view of (9) and (10), $\left|\frac{g}{2}\right| \neq 1$. Hence it must be that $\left|\frac{g}{2}\right| < 1$. Let

$$a = -\frac{g}{2} + i\sqrt{1 - \left(\frac{g}{2}\right)^2} = p + iq = e^{iq}, \quad b = p - iq = e^{-iq},$$

where

$$p = -\frac{g}{2}, \quad q = \sqrt{1 - \left(\frac{g}{2}\right)^2} \neq 0,$$

and

$$q = \tan^{-1}\frac{q}{p}, \quad \sin q = q \neq 0, \quad \cos q = p \neq 1.$$

Then

$$a^L - b^L = e^{iLq} - e^{-iLq} = 2i \sin Lq, \tag{14}$$

$$a - b = 2i \sin q, \tag{15}$$

and $\frac{a^L - b^L}{a-b} = 0$ implies

$$Lq = np, \; q_n = \frac{np}{L}, \; n = \pm 1, 2, \ldots, \pm(L-1). \tag{16}$$

Substituting (14) ~ (16) into (13), we get the possible solutions:

$$U_k^{(t)} = A_1 + \frac{B_1 - A_1}{L}k + l_n^t a_0 b_{1n} \frac{\sin kq_n}{\sin q_n},$$
$$n = \pm 1, \pm 2, \ldots, \pm(L-1); \; k = 0, 1, 2, \ldots, L$$

where

$$l_n = 1 - 2d\left(1 - \cos q_n\right). \tag{17}$$

By superposition, we obtain the general solution

$$
\begin{aligned}
U_k^{(t)} &= A_1 + \frac{B_1 - A_1}{L} k + 2a_0 \sum_{n=1}^{L-1} l_n^t b_{1n} \frac{\sin \frac{knp}{L}}{\sin \frac{np}{L}} \\
&= A_1 + \frac{B_1 - A_1}{L} k + \sum_{n=1}^{L-1} l_n^t D_n \sin \frac{knp}{L} \qquad (18) \\
& k = 0, 1, 2, \ldots, L; \ t = 0, 1, 2, \ldots ,
\end{aligned}
$$

where

$$
D_n = \frac{2a_0 b_{1n}}{\sin \frac{np}{L}}, \ n = 1, 2, \ldots, (L-1)
$$

are $L-1$ coefficients to be determined by the initial conditions of the system.

Suppose the initial condition is given by $U_k^{(0)} = g(k)$, $k = 0, 1, 2, \ldots, L$, then

$$
U_k^{(0)} = A_1 + C_1 k + \sum_{n=1}^{L-1} D_n \sin \frac{knp}{L} = g(k).
$$

Hence

$$
\sum_{n=1}^{L-1} D_n \sin \frac{knp}{L} = g(k) - (A_1 + C_1 k) = f(k), \qquad (19)
$$

and D_n can be determined by the orthogonality formulas that will be derived in the next section.

We remark that since the boundary conditions always demand $b_0 = 0$, by (11), we have

$$
b = \frac{1}{a-b} \left\{ b_1 \left(a^k - b^k \right) \right\}_{k=0}^{\infty} .
$$

Then the requirement

$$
\frac{a^L - b^L}{a-b} = 0
$$

is equivalent to $b_L = 0$.

We now return to (8),

$$
db_{k-1} + (1-2d) b_k + db_{k+1} = lb_k, \ k = 1, 2, \ldots, L-1 .
$$

This may be written in form of a system of equations

$$
\begin{aligned}
db_0 + (1-2d) b_1 + db_2 &= lb_1 \\
db_1 + (1-2d) b_2 + db_3 &= lb_2 \\
\cdots &= \cdots \\
db_{L-2} + (1-2d) b_{L-1} + db_L &= lb_{L-1} .
\end{aligned}
$$

In case $b_0 = b_L = 0$, the above system may also be written as

$$
Ab = lb ,
$$

where

$$A=\begin{pmatrix} (1-2d) & d & 0 & 0 & \cdots & \cdots & 0 & 0 \\ d & (1-2d) & d & 0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & d & (1-2d) & d \\ 0 & 0 & 0 & 0 & \cdots & 0 & d & (1-2d) \end{pmatrix}_{L-1\times L-1}, \tag{20}$$

$$b=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{L-1} \end{pmatrix}.$$

The eigenvalue of A is known to be [4]

$$l_n=(1-2d)+2d\cos\frac{np}{L}=1-2d\left(1-\cos\frac{np}{L}\right),$$

which is same as (17), with corresponding eigenvectors

$$b_n=col\left(\sin\frac{np}{L},\sin\frac{2np}{L},\cdots,\sin\frac{(L-1)np}{L}\right),\quad n=1,2,\ldots,L-1, \tag{21}$$

so that the k-th term in b_n (corresponding to n-th eigenvector) is

$$b_{n,k}=\sin\frac{knp}{L},\quad k=1,2,\ldots,L-1.$$

The solution of (1) is then

$$U_k^{(t)}=a_t b_k=l_n^t a_0\sin\frac{knp}{L},\quad n=1,2,\ldots,L-1.$$

By superposition, we obtain the same results as (18)

$$U_k^{(t)}=A_1+\frac{B_1-A_1}{L}k+\sum_{n=1}^{L-1}l_n^t D_n\sin\frac{knp}{L},\quad k=0,1,2,...,L.\quad t=0,1,2,\ldots.$$

3. Orthogonality formulas of the eigenvectors

Before we can solve any difference equation arising from actual heat transfer problem by means of (18) and (19), we need some auxiliary formulas that concern the orthogonality properties of the eigenvectors in (21). Some of these formulas may be known in the literature, but are included here for the sake of completeness.

Formula 1

$$\sum_{k=1}^{L-1}\cos\frac{kmp}{L}=\begin{cases}0 & m\ odd,\\ -1 & m\ even,\end{cases}$$

$$\sum_{k=1}^{L-1}\sin\frac{kmp}{L}=\begin{cases}\cot\frac{mp}{2L} & m\ odd,\\ 0 & m\ even.\end{cases}$$

Proof: Since $De^{ikq}=e^{i(k+1)q}-e^{ikq}=e^{ikq}\left(e^{iq}-1\right)$, we have

$$\sum_{k=1}^{L-1}e^{ikq}=\frac{1}{e^{iq}-1}\sum_{k=1}^{L-1}De^{ikq}=\frac{e^{iLq}-e^{iq}}{e^{iq}-1}. \tag{22}$$

Let $q=\frac{mp}{L}$, and note that $e^{iLq}=e^{imp}=(-1)^m$, we see that

$$\sum_{k=1}^{L-1} e^{ikq}=\frac{-\left(e^{iq}-(-1)^m\right)}{e^{iq}-1}=\frac{-\left(e^{i\frac{q}{2}}-(-1)^m e^{-i\frac{q}{2}}\right)}{e^{i\frac{q}{2}}-e^{-i\frac{q}{2}}}.$$

When m is odd, it becomes

$$\sum_{k=1}^{L-1} e^{ikq}=\frac{-2\cos\frac{q}{2}}{2i\sin\frac{q}{2}}=i\cot\frac{q}{2},$$

and when m is even,

$$\sum_{k=1}^{L-1} e^{ikq}=-1.$$

Hence

$$\sum_{k=1}^{L-1}\left(\cos\frac{kmp}{L}+i\sin\frac{kmp}{L}\right)=\begin{cases} i\cot\frac{mp}{2L} & m \ odd, \\ -1 & m \ even. \end{cases}$$

By equating the real and imaginary parts separately, we obtain the desired results.

Formula 2.

$$\sum_{k=1}^{L-1}\sin\frac{knp}{L}\sin\frac{kmp}{L}=\begin{cases} 0 & n\neq m, \\ \frac{L}{2} & n=m. \end{cases}$$

Proof: Since

$$\sum_{k=1}^{L-1}\sin\frac{knp}{L}\sin\frac{kmp}{L}=\frac{1}{2}\cdot\sum_{k=1}^{L-1}\left(\cos\frac{k(n-m)p}{L}-\cos\frac{k(n+m)p}{L}\right),$$

we see that for $n\neq m$,

$$\sum_{k=1}^{L-1}\sin\frac{knp}{L}\sin\frac{kmp}{L}=0.$$

For $n=m$,

$$\sum_{k=1}^{L-1}\sin^2\left(\frac{kmp}{L}\right)=\frac{1}{2}\sum_{k=1}^{L-1}\left(1-\cos\frac{2kmp}{L}\right)=\frac{1}{2}(L-1)-\frac{1}{2}(-1)=\frac{L}{2}.$$

This completes the proof.

As a direct consequence, we have the following

Formula 3.

$$\sum_{n=1}^{L-1}\sum_{k=1}^{L-1} D_n\sin\frac{kmp}{L}\sin\frac{knp}{L}=\frac{L}{2}D_m.$$

Formula 4.

$$\sum_{k=1}^{L-1} k\cos\left(\frac{kmp}{L}\right)=\begin{cases}\dfrac{L}{2}-\dfrac{1}{2}\csc^2\dfrac{mp}{2L} & m\ odd,\\ -\dfrac{L}{2} & m\ even,\end{cases}$$

$$\sum_{k=1}^{L-1} k\sin\left(\frac{kmp}{L}\right)=\begin{cases}\dfrac{L}{2}\cot\dfrac{mp}{2L} & m\ odd,\\ -\dfrac{L}{2}\cot\dfrac{mp}{2L} & m\ even.\end{cases}$$

Proof: By (22), let $h(q)=\sum_{k=1}^{L-1} e^{ikq}=\dfrac{e^{iLq}-e^{iq}}{e^{iq}-1}$, we have

$$h'(q)=\sum_{k=1}^{L-1} ike^{ikq}=\frac{iLe^{iLq}-ie^{iq}}{e^{iq}-1}-\frac{ie^{iq}\left(e^{iLq}-e^{iq}\right)}{\left(e^{iq}-1\right)^2}.$$

Hence

$$\sum_{k=1}^{L-1} ke^{ikq}=\frac{Le^{iLq}}{e^{iq}-1}-\left(\frac{e^{iq}}{e^{iq}-1}+\frac{e^{iq}\left(e^{iLq}-e^{iq}\right)}{\left(e^{iq}-1\right)^2}\right)$$

$$=\frac{Le^{iLq}}{e^{iq}-1}+\frac{e^{iq}\left(1-e^{iLq}\right)}{\left(e^{iq}-1\right)^2}. \tag{23}$$

Let $q=\dfrac{mp}{L}$, we see that when m is odd,

$$\sum_{k=1}^{L-1} ke^{ikq}=\frac{-L}{e^{iq}-1}+\frac{2e^{iq}}{\left(e^{iq}-1\right)^2}$$

$$=\frac{-Le^{-i\frac{q}{2}}}{e^{i\frac{q}{2}}-e^{-i\frac{q}{2}}}+\frac{2}{\left(e^{i\frac{q}{2}}-e^{-i\frac{q}{2}}\right)^2}$$

$$=\frac{-L\left(\cos\dfrac{mp}{2L}-i\sin\dfrac{mp}{2L}\right)}{2i\sin\dfrac{mp}{2L}}+\frac{1}{-2\sin^2\dfrac{mp}{2L}}$$

$$=\frac{L}{2}-\frac{1}{2}\csc^2\frac{mp}{2L}+i\frac{L}{2}\cot\frac{mp}{2L},$$

and when m is even,

$$\sum_{k=1}^{L-1} ke^{ikq}=\frac{L}{e^{iq}-1}=-\frac{L}{2}-i\frac{L}{2}\cot\frac{mp}{2L}.$$

In summary,

$$\sum_{k=1}^{L-1} k\left(\cos\frac{kmp}{L}+i\sin\frac{kmp}{L}\right)=\begin{cases}\dfrac{L}{2}-\dfrac{1}{2}\csc^2\dfrac{mp}{2L}+i\dfrac{L}{2}\cot\dfrac{mp}{2L} & m\ odd,\\ -\dfrac{L}{2}-i\dfrac{L}{2}\cot\dfrac{mp}{2L} & m\ even.\end{cases}$$

By equating the real and imaginary parts separately, we obtain the desired results.

Formula 5.

$$\sum_{k=1}^{L-1} k^2\cos\frac{kmp}{L}=\begin{cases}\dfrac{L^2}{2}-\dfrac{L}{2}\csc^2\dfrac{mp}{2L} & m\ odd,\\ -\dfrac{L^2}{2}+\dfrac{L}{2}\csc^2\dfrac{mp}{2L} & m\ even,\end{cases}$$

$$\sum_{k=1}^{L-1} k^2\sin\frac{kmp}{L}=\begin{cases}\dfrac{1}{2}\cot\dfrac{mp}{2L}\left(L^2-\csc^2\dfrac{mp}{2L}\right) & m\ odd,\\ -\dfrac{1}{2}L^2\cot\dfrac{mp}{2L} & m\ even.\end{cases}$$

Proof: Let $g(q)=\sum_{k=1}^{L-1} ke^{ikq}$, then $g'(q)=\sum_{k=1}^{L-1} ik^2e^{ikq}$. By (23),

$$g(q)=\frac{Le^{iLq}}{e^{iq}-1}+\frac{e^{iq}\left(1-e^{iLq}\right)}{\left(e^{iq}-1\right)^2},$$

we have

$$g'(q)=\frac{iL^2e^{iLq}}{e^{iq}-1}-\frac{ie^{iq}Le^{iLq}}{\left(e^{iq}-1\right)^2}$$
$$+\frac{ie^{iq}\left(1-e^{iLq}\right)-iLe^{iLq}e^{iq}}{\left(e^{iq}-1\right)^2}-\frac{2ie^{iq}e^{iq}\left(1-e^{iLq}\right)}{\left(e^{iq}-1\right)^3}.$$

Hence

$$\sum_{k=1}^{L-1} k^2e^{ikq}=\frac{L^2e^{iLq}}{e^{iq}-1}+\frac{e^{iq}\left(1-e^{iLq}\right)-2Le^{iLq}e^{iq}}{\left(e^{iq}-1\right)^2}-\frac{2e^{i2q}\left(1-e^{iLq}\right)}{\left(e^{iq}-1\right)^3}.$$

Let $q=\dfrac{mp}{L}$, we see that when m is odd,

$$\sum_{k=1}^{L-1} k^2e^{ikq}=\frac{-L^2}{e^{iq}-1}+\frac{2e^{iq}+2Le^{iq}}{\left(e^{iq}-1\right)^2}-\frac{4e^{i2q}}{\left(e^{iq}-1\right)^3}$$

$$=\frac{-L^2e^{-i\frac{q}{2}}}{e^{i\frac{q}{2}}-e^{-i\frac{q}{2}}}+\frac{2(L+1)}{\left(e^{i\frac{q}{2}}-e^{-i\frac{q}{2}}\right)^2}-\frac{4e^{i\frac{q}{2}}}{\left(e^{i\frac{q}{2}}-e^{-i\frac{q}{2}}\right)^3}$$

$$=\frac{-L^2\left(\cos\frac{mp}{2L}-i\sin\frac{mp}{2L}\right)}{2i\sin\frac{mp}{2L}}+\frac{L+1}{-2\sin^2\frac{mp}{2L}}-\frac{\left(\cos\frac{mp}{2L}+i\sin\frac{mp}{2L}\right)}{-2i\sin^3\frac{mp}{2L}}$$

$$= \frac{L^2}{2} - \frac{L}{2}\csc^2\frac{mp}{2L} + i\frac{1}{2}\cot\frac{mp}{2L}\left(L^2 - \csc^2\frac{mp}{2L}\right),$$

and when m is even,

$$\sum_{k=1}^{L-1} k^2 e^{ikq} = \frac{L^2}{e^{iq}-1} + \frac{-2Le^{iq}}{\left(e^{iq}-1\right)^2}$$

$$= \frac{L^2\left(\cos\frac{mp}{2L} - i\sin\frac{mp}{2L}\right)}{2i\sin\frac{mp}{2L}} + \frac{L}{2\sin^2\frac{mp}{2L}}$$

$$= -\frac{L^2}{2} + \frac{L}{2}\csc^2\frac{mp}{2L} - i\frac{L^2}{2}\cot\frac{mp}{2L}.$$

In summary,

$$\sum_{k=1}^{L-1} k^2\left(\cos\frac{kmp}{L} + i\sin\frac{kmp}{L}\right) = \begin{cases} \frac{L^2}{2} - \frac{L}{2}\csc^2\frac{mp}{2L} + i\frac{1}{2}\cot\frac{mp}{2L}\left(L^2 - \csc^2\frac{mp}{2L}\right) & m\ odd, \\ -\frac{L^2}{2} + \frac{L}{2}\csc^2\frac{mp}{2L} - i\frac{L^2}{2}\cot\frac{mp}{2L} & m\ even. \end{cases}$$

By equating the real and imaginary parts separately, we obtain the desired results.

4. Solutions and Examples

The results of the preceding sections may be summarized into the following theorems.

Theorem 1. The solution of

$$U_k^{(t+1)} - U_k^{(t)} = d\left(U_{k-1}^{(t)} - 2U_k^{(t)} + U_{k+1}^{(t)}\right) \tag{24}$$

is given by

$$U_k^{(t)} = A_1 + \frac{B_1 - A_1}{L}k + \sum_{n=1}^{L-1} l_n^t D_n \sin\frac{knp}{L}, \quad k = 0, 1, 2, ..., L\ ;\ \ t = 0, 1, 2, \ldots\ , \tag{25}$$

where

$$l_n = 1 - 2d\left(1 - \cos\frac{np}{L}\right), \quad n = 1, 2, \ldots, L-1. \tag{26}$$

$A_1 = U_0^{(t)}$ and $B_1 = U_L^{(t)}$ are the boundary values of the system. The coefficients D_n may be obtained by

$$D_m = \frac{2}{L}\sum_{k=1}^{L-1}\sin\frac{kmp}{L}\ f(k); \quad m = 1, 2, \ldots, L-1, \tag{27}$$

where

$$f(k) = g(k) - (A_1 + C_1 k) \tag{28}$$

and $g(k) = U_k^{(0)}$ is the initial value of the system.

Theorem 2. The terminal values of b (that is, b_0 and b_L) in the equation

$$db_{k-1} + (1-2d)b_k + db_{k+1} = lb_k, \quad k = 1, 2, \ldots, L-1$$

are always equal to 0 regardless of the boundary conditions.

Example 1: Suppose that a thin metal rod 1 meter long with insulated lateral surface has uniform initial temperature $U_k^{(0)} = 40$. At time $t \geq 0$, the left end (set as the origin of the coordinates) is in contact with a heat source of $U = 30$ and the right end is in contact with a heat source of $U = 50$. We will treat our rod as a lumped rod and try to find the temperature $U_k^{(t)}$ for $t \geq 0$, where we take every 1 cm of the length L as a testing point k (thus, $k = 0, 1, 2, ..., 100$).

In our example, we suppose $d = 0.25$.

Solution: The difference equation is given by

$$U_k^{(t+1)} - U_k^{(t)} = 0.25\left(U_{k-1}^{(t)} - 2U_k^{(t)} + U_{k+1}^{(t)}\right), \quad k = 0, 1, 2, ..., 100 \; ; \quad t = 0, 1, 2, \ldots \quad .$$

And the solution, by (25), after substituting the boundary conditions, is found to be

$$U_k^{(t)} = 30 + 0.2k + \sum_{n=1}^{99} l_n^t D_n \sin\frac{knp}{100}, \quad k = 0, 1, 2, ..., 100 \; ; \quad t = 0, 1, 2, \ldots$$

where $l_n = 1 - 0.5\left(1 - \cos\frac{np}{100}\right) = 0.5\left(1 + \cos\frac{np}{100}\right), \quad n = 1, 2, \ldots, 99$. At $t = 0$, we have

$$30 + 0.2k + \sum_{n=1}^{99} D_n \sin\frac{knp}{100} = 40,$$

hence

$$\sum_{n=1}^{99} D_n \sin\frac{kn\pi}{100} = 10 - 0.2k = f(k).$$

Applying (27) and formulas 1 and 4,

$$D_m = \frac{2}{100}\sum_{k=1}^{99}(10 - 0.2k)\sin\frac{kmp}{100}$$

$$= 0.02 \times \begin{cases} 10\cot\frac{mp}{200} - 0.2 \cdot \frac{100}{2}\cot\frac{mp}{200} = 0 & m \ odd, \\ 0.2 \cdot \frac{100}{2}\cot\frac{mp}{200} & m \ even, \end{cases}$$

or

$$D_m = \begin{cases} 0 & m \ odd, \\ 0.2\cot\frac{mp}{200} & m \ even. \end{cases}$$

Hence the total solution becomes

$$U_k^{(t)} = 30 + 0.2k + 0.2\sum_{n \ even}^{99}\left(0.5\left(1 + \cos\frac{np}{100}\right)\right)^t \cot\frac{np}{200}\sin\frac{knp}{100},$$

$$k = 0, 1, 2, ..., 100 \; ; \quad t = 0, 1, 2, \ldots \quad .$$

Example 2: If initially the left end of the metal rod in the above problem is in contact with a heat source of $U = 90$ and the right end is in contact with a heat source of $U = 10$ for a very long time (so that it has been reached the thermal equilibrium or steady state), rework the problem with the remaining situations unchanged.

Solution: Before $t = 0$, the system is in steady state, so it satisfies the equation

$$U_k^{(t)} = A_0 + C_0 k \,.$$

Then with $U_0^{(t)} = A_0 = 90$ and $U_{100}^{(t)} = 90 + 100C_0 = 10$, we have $C_0 = -0.8$. The initial condition is then $U_k^{(0)} = 90 - 0.8k = g(k)$. Hence at $t = 0$, we have

$$30 + 0.2k + \sum_{n=1}^{99} D_n \sin\frac{knp}{100} = 90 - 0.8k \,,$$

or

$$\sum_{n=1}^{99} D_n \sin\frac{knp}{100} = 60 - k = f(k) \,.$$

Applying (27),

$$D_m = \frac{2}{100}\sum_{n=1}^{99}(60-k)\sin\frac{kmp}{100}$$

$$= 0.02 \times \begin{cases} 60\cot\dfrac{mp}{200} - \dfrac{100}{2}\cot\dfrac{mp}{200} = 10\cot\dfrac{mp}{200} & m \ odd, \\ \dfrac{100}{2}\cot\dfrac{mp}{200} & m \ even, \end{cases}$$

or

$$D_m = \begin{cases} 0.2\cot\dfrac{mp}{200} & m \ odd, \\ \cot\dfrac{mp}{200} & m \ even. \end{cases}$$

Hence the total solution becomes

$$U_k^{(t)} = 30 + 0.2k + 0.2\sum_{n\ odd}^{99}\left(0.5\left(1+\cos\frac{np}{100}\right)\right)^t \cot\frac{np}{200}\sin\frac{knp}{100}$$

$$+ \sum_{n\ even}^{99}\left(0.5\left(1+\cos\frac{np}{100}\right)\right)^t \cot\frac{np}{200}\sin\frac{knp}{100} \,,$$

$$k = 0, 1, 2, ..., 100 \,; \quad t = 0, 1, 2, \ldots \,.$$

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