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A new process capability index for non-normal distributions

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Abstract *Many process capability indices have been proposed to measure process performance. In this paper, we first review C_p, C_{pk}, C_{pm} and C_{pmk}, and their generalizations, C_{Np}, C_{Npk}, C_{Npm} and C_{Npmk}, and then propose a new index S_{pmk} for any underlying distribution, which takes into account process variability, departure of the process mean from the target value, and proportion of nonconformity. Proportion of nonconformity can be exactly reflected by S_{pmk}. Its superiority over C_{Npmk}, a recently developed index, also taking into account process variability and departure from the target value, is demonstrated with several non-normal processes. A method is proposed to estimate S_{pmk}, with illustrations.*

Introduction

Many process capability indices have been proposed to provide numerical measures on process performance. They have been widely used in the manufacturing industry in Japan and the USA. Kane (1986) discussed two commonly used indices C_p and C_{pk}. Chan *et al.* (1988a) and Pearn *et al.* (1992) developed two more-advanced indices C_{pm} and C_{pmk}. Choi and Owen (1990) gave detailed discussions and comparisons for C_p, C_{pk}, C_{pm} and C_{pmk}. Boyles (1994) proposed the capability indices with asymmetric tolerance for C_{pk}, C_{pm} and C_{pmk}. Discussions and analysis of these indices on point estimation and interval estimation have been the focus of many statisticians and quality researchers including Kane (1986), Chan *et al.* (1988a), Chou *et al.* (1990), Pearn *et al.* (1992), Kotz *et al.* (1993), Vännman (1995), Pearn and Chen (1996), and many others. These investigations, however, are based on the assumption that the process underlying distribution is normal. If the assumption is not satisfied, then these basic indices are unreliable (Chan *et al.*, 1988b; Gunter, 1989a; 1989b; English and Taylor, 1993; Somerville and Montgomery, 1997; Chen and Pearn, 1997). Zwick (1995), Schneider *et al.* (1995), Pearn and Chen (1995), Chen and Pearn (1997), Tong and Chen (1998) gave process capability indices for non-normal distributions. In this paper, a new process capability index S_{pmk} for non-normal distributions is proposed. The index can exactly reflect proportion of nonconformity.

The $C_p(u,v)$ and $C_{np}(u,v)$ indices

Vännman (1995) defined a class of capability indices, depending on two non-negative parameters, u and v, as:

$$C_p(u,v) = \frac{d - u|\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \quad (1)$$

where μ is the process mean, σ the process standard deviation, $d = (USL - LSL)/2$, where *USL* and *LSL* are respectively the upper and lower specification limits, $m = (USL + LSL)/2$, the specification center, and T is the target value. The four basic indices, C_p, C_{pk}, C_{pm} and C_{pmk} are special cases of $C_p(u, v)$ by letting $u = 0$ or 1 and $v = 0$ or 1. More specifically, $C_p(0, 0) = C_p$, $C_p(1, 0) = C_{pk}$, $C_p(0, 1) = C_{pm}$, and $C_p(1, 1) = C_{pmk}$, where:

$$C_p = \frac{USL - LSL}{6\sigma}, \quad (2)$$

$$C_{pk} = \frac{\min\{USL - \mu, \mu - LSL\}}{3\sigma}, \quad (3)$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \quad (4)$$

$$C_{pmk} = \frac{\min\{USL - \mu, \mu - LSL\}}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \quad (5)$$

C_p takes into account the process variance only. It cannot detect departure of the process mean from the specification center, and therefore cannot be used to fully evaluate process capability. C_{pk} takes into account both the process mean and the process variance. When $\mu = m$, $C_{pk} = C_p$. If $\mu \neq m$, $C_{pk} = C_p\,(1 - K)$, where $K = 2 \mid \mu - m \mid / (USL - LSL)$, $LSL \leq \mu \leq USL$. However, C_{pk} cannot detect departure from the target value. Although C_{pm} overcomes this drawback by taking into account $(\mu - T)^2$, it cannot detect the location of the process mean in the interval (*LSL*, *USL*) (Choi and Owen, 1990). C_{pmk} takes into account process variability, departure from the target value and location of the process mean in (*LSL*, *USL*). When $\mu = m$, $C_{pmk} = C_{pm}$; If $\mu \neq m$, $C_{pmk} = C_{pm}\,(1 - K)$. When the process distribution is normal and $\mu = m$, there exists a one-to-one relationship between C_p (= C_{pk}) and proportion of nonconformity P, given by $3C_p = \Phi^{-1}(1 - P/2)$, where Φ denotes the *CDF* of the standard normal distribution. For example, when $C_p = 1$, $P = 0.27$ per cent; when $C_p = 1.33$, $P = 0.0066$ per cent. When $\mu \neq m$, there is no one-to-one relationship between C_{pk} and P. Given a C_{pk} value, Boyles (1991) gave an upper bound $\Phi(3C_{pk})$ and a lower bound $(2\Phi 3C_{pk} - 1)$ for proportion of conformity. Boyles (1994) further proposed a smoothing index S_{pk} to setup a one-to-one relationship with P. That is, $3S_{pk} = \Phi^{-1}(1 - P/2)$. It follows that evaluation of process capability can be based on the following three criteria:

(1) variability in process;

(2) degree of departure of the process mean from the target value;

(3) location of the process mean in the interval (*LSL*, *USL*).

C_p takes into account criterion (1) only, C_{pk} criteria (1) and (3), C_{pm} criteria (1) and (2), and C_{pmk} criteria (1), (2) and (3). The larger a capability index value for a process is, the more capable the process is. However, as mentioned before, when the process underlying distribution is non-normal, these indices may not be appropriate to evaluate process capability. Although Zwick (1995) and Schneider *et al.* (1995) provided capability indices for any distribution (normal or non-normal), their performances were not evaluated. Chen and Pearn (1997) and Tong and Chen (1998) proposed generalizations of $C_p(u, v)$ for any underlying distribution as follows:

$$C_{Np}(u,v) = \frac{d - u|M - m|}{3\sqrt{\left(\frac{F_{99.865} - F_{0.135}}{6}\right)^2 + V(M - T)^2}} \quad (6)$$

where F_α is the (100 α)th percentile of the distribution and M is the median. Note that the generalizations were developed by replacing μ in (1) by M, and σ by $(F_{99.865} - F_{0.135})/6$. Setting $(u, v) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$ leads to the four basic indices for any underlying distribution, referred to as C_{Np}, C_{Npk}, C_{Npm} and C_{Npmk}:

$$C_{Np} = \frac{USL - LSL}{(F_{99.865} - F_{0.135})} \quad (7)$$

$$C_{Npk} = \frac{\min\{USL - M, M - LSL\}}{\left(\frac{F_{99.865} - F_{0.135}}{2}\right)} \quad (8)$$

$$C_{Npm} = \frac{USL - LSL}{6\sqrt{\left(\frac{F_{99.865} - F_{0.135}}{6}\right)^2 + (M - T)^2}} \quad (9)$$

$$C_{Npmk} = \frac{\min\{USL - M, M - LSL\}}{3\sqrt{\left(\frac{F_{99.865} - F_{0.135}}{6}\right)^2 + (M - T)^2}}. \quad (10)$$

In addition, if M in (6) is replaced by μ, Chen and Pearn (1997) called the indices $C'_{Np}(u, v)$. For the normal case, $M = \mu$, $(F_{99.865} - F_{0.135})/6 = \sigma$, and therefore $C_{Np}(u, v) = C'_{Np}(u, v) = C_p(u, v)$. Chen and Pearn (1997) indicated that, if the process underlying distribution is chi-square with three degrees of freedom, then proportion of nonconformity can be reflected by $C_{Np}(u, v)$ better than $C_p(u, v)$ and $C'_{Np}(u, v)$. In this paper, a new process capability index S_{pmk}, taking into account process variability, departure of the process mean from the target value, and proportion of nonconformity, will be proposed for any underlying distribution. As will be seen later, proportion of nonconformity can be exactly reflected by S_{pmk}. Its superiority over C_{Npmk}, also taking into account process variability and departure from the target value, will be illustrated for several non-normal distributions including chi-square, gamma, exponential, and uniform.

A new index S_{pmk}

A new process capability index, S_{pmk}, is proposed for any underlying distribution as follows:

$$S_{pmk} = \frac{\Phi^{-1}(\frac{1+F(USL)-F(LSL)}{2})}{3\sqrt{1+(\frac{\mu-T}{\sigma})^2}} = \frac{\Phi^{-1}(1-P/2)}{3\sqrt{1+(\frac{\mu-T}{\sigma})^2}} \quad (11)$$

where $F(x)$ denotes the *CDF* of the process distribution. The idea comes from the properties that:

$$C_{pmk} = C_{pm}(1-K) = \frac{C_p}{\sqrt{1+\left(\frac{\mu-T}{\sigma}\right)^2}}(1-K) = \frac{C_{pk}}{\sqrt{1+\left(\frac{\mu-T}{\sigma}\right)^2}}, \quad (12)$$

and C_{pk} can be substituted, under any distribution, with $\frac{1}{3}\Phi^{-1}(\frac{1+F(USL)-F(LSL)}{2})$ (Chen, 2000). Proportion of nonconformity P can be exactly evaluated through S_{pmk} by $2 \times (1-\Phi(3S_{pmk}\sqrt{1+(\frac{\mu-T}{\sigma})^2})$. Note that if the process underlying distribution is normal and $\mu = m$, then $S_{pmk} = C_{Npmk} = C_{pmk} = C_{pm}$.

Comparisons

In this section, we will demonstrate the superiority of S_{pmk} over C_{Npmk} with six non-normal processes. Let $\chi^2{}_3$ denote a chi-square distribution with three degrees of freedom. Let the underlying distribution of process A be $\chi^2{}_3 + 7$, that of process B be $\chi^2{}_3 + 14.8$, and that of process C be $\chi^2{}_3 + 22.6$, as in Chen and Pearn (1997). In addition, let process D have a gamma distribution with parameters $\alpha = 6$, $\beta = 3$, process E have a gamma distribution with parameters $\alpha = 1$, $\beta = 12$, i.e. an exponential distribution, and process F have a uniform (17, 25.8) distribution. The distribution characteristics are summarized in Table I.

The target values for the processes are all 17.8, and *USL* and *LSL* are respectively 10.0 and 25.6. Since S_{pmk} can exactly reflect the actual proportion of nonconformity P (= $P(X < 10.0) + P(X > 25.6)$), the difference between P and $2 \times (1-\Phi(3S_{pmk}\sqrt{1+(\frac{\mu-T}{\sigma})^2})$ (proportion of nonconformity obtained through

S_{pmk}) is zero. On the other hand, to see how well C_{Npmk} can reflect the actual proportion of nonconformity, simply examine the relative error size $ERR_{CNpmk} = |P'_{CNpmk} - P| / P$, where P'_{CNpmk} denotes the predicted proportion of nonconformity associated with C_{Npmk}, and can be obtained by $P'_{CNpmk} = 2 \times (1 - \Phi(3C_{Npmk}\sqrt{1 + (\frac{M-T}{(F_{99.865}-F_{0.135})/6})^2})$. Table II displays the values of P, S_{pmk}, C_{Npmk}, P'_{CNpmk} and ERR_{CNpmk} for the six processes. Computations were carried out by using the SAS software (SAS Institute, Inc., 1990). The results show that values of ERR_{CNpmk} are remarkable for all of the cases, indicating the deficiency of the C_{Npmk} index.

An estimator of S_{pmk}

When the process has a distribution of Pearsonian type, Pearn and Chen (1995) proposed an estimator for $C_{Np}(u, v)$ by using Clements' method (Clements, 1989) as follows:

$$\hat{C}_{Np}(u,v) = \frac{d - u\left|\hat{M} - m\right|}{3\sqrt{\left(\frac{U_p - L_p}{6}\right)^2 + v(\hat{M} - T)^2}}, \tag{13}$$

Process	Underlying distribution	μ	M	σ	$F_{0.135}$	$F_{99.865}$
A	$\chi^2_3 + 7$	10.00	9.366	2.45	7.030	22.630
B	$\chi^2_3 + 14.8$	17.80	17.166	2.45	14.830	30.430
C	$\chi^2_3 + 22.6$	25.60	24.966	2.45	22.630	38.230
D	Gamma (6, 3)	18.00	17.010	7.35	3.525	48.104
E	Gamma (1, 12)	12.00	8.318	12.00	0.016	79.292
F	Uniform (17, 25.8)	21.40	21.400	2.54	17.012	25.788

Table I. Distribution characteristics of the six processes

Process	P	S_{pmk}	C_{Npmk}	P'_{CNpmk}	ERR_{CNpmk} (%)
A	0.6087	0.0511	0.0000	1.0000	64.3
B	0.0129	0.8292	0.8925	0.0059	54.3
C	0.3916	0.0856	0.0277	0.8074	106.2
D	0.2683	0.3689	0.3128	0.3454	28.7
E	0.7571	0.0928	0.0000	1.0000	32.1
F	0.0227	0.4378	0.3603	0.0041	81.9

Table II. Comparisons between S_{pmk} and C_{Npmk}

where U_p is an estimator for $F_{99.865}$, L_p for $F_{0.135}$, and $\hat{M}$ for median M, and their estimates can be obtained by using the tables developed by Gruska *et al.* (1989). In practice, the process underlying distribution is always unknown. Chang and Lu (1994) compute estimates for $F_{99.865}$, $F_{0.135}$, and M based on sample percentiles instead of Gruska *et al.*'s (1989) tables. An estimator for S_{pmk} follows:

$$\hat{S}_{pmk} = \frac{\Phi^{-1}\left(\frac{1+\hat{F}(USL)-\hat{F}(LSL)}{2}\right)}{3\sqrt{1+\left(\frac{\bar{X}-T}{S}\right)^2}} = \frac{\Phi^{-1}(1-\hat{P}/2)}{3\sqrt{1+\left(\frac{\bar{X}-T}{S}\right)^2}}, \qquad (14)$$

where $\hat{F}$ (*USL*) denotes the sample proportion of those less than or equal to *USL*, $\hat{F}$ (LSL) the sample proportion of those less than *LSL*, $\bar{X}$ the sample mean, S the sample standard deviation, and $\hat{P}$ the sample proportion of nonconformity. Note that the estimator in expression (14) may not perform well for small samples.

An example

To illustrate how to calculate $\hat{S}_{pmk}$, we use data provided by the Tung Pei Industrial Co. Ltd, a manufacturer of bearings in Taiwan. The bearings manufactured are certified by CNS, JIS and ISO. The number of bearings produced is about ten million. Outer ring, inner ring, rolling body, and retainer are four main components of roller bearing. Items for examining roller bearing include size variation, dimension variation, and rotation tolerance. In this illustration, only single row deep groove radial ball bearing (index number 6212) is discussed (Figure 1), and air gauge or cylinder gauge is used to measure the size variation. Let d denote diameter of the inner ring and its target value be 60mm. According to the CNS2862 standard, *USL* for *d* is 60.004mm and *LSL* is 59.981mm. If *d* falls outside specification limits, it is unacceptable. Table III displays a random sample of size 100 for *d*. Normality has been examined by using the Shapiro and Wilk (1965) *W* test. Using the SAS

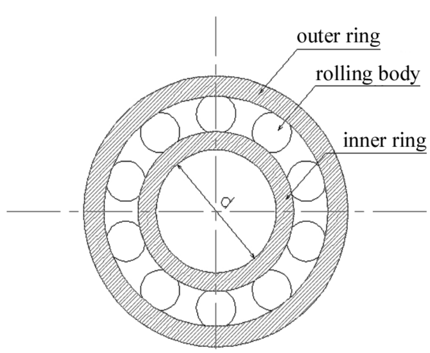

Figure 1.
The structure of ball bearing

Table III.
100 observations of inner diameter for roller bearing

59.984	59.981	59.981	60.003	59.982	60.005	60.004	59.983	59.981	59.980
60.000	59.998	59.982	59.983	59.981	59.982	59.999	60.001	59.982	59.988
59.995	59.998	59.982	59.983	59.981	59.994	60.002	59.988	59.980	59.982
59.982	59.983	59.981	59.986	59.987	60.001	59.982	60.003	60.001	59.984
59.985	59.979	59.987	59.990	59.998	59.984	59.989	59.999	59.985	60.003
60.004	60.001	60.000	59.982	59.981	59.984	59.998	59.983	59.999	59.987
59.991	59.992	59.992	59.983	59.981	59.996	59.997	60.000	60.000	59.991
60.002	60.001	59.990	59.987	59.982	60.006	59.981	59.982	59.984	59.985
60.003	60.004	59.992	59.991	59.986	59.992	59.991	59.981	59.998	59.985
60.001	59.980	59.993	59.984	59.981	59.984	59.988	59.999	60.000	60.001

software, we have $W = 0.8618$ with p-value < 0.0001. Since the p-value is very small, we conclude that the data were drawn from a non-normal distribution. To find $\hat{S}_{pmk}$, we first calculate $\hat{F}(USL) = 98/100$, $\hat{F}(LSL) = 4/100$, $\bar{X} = 59.9903$, and $S = 0.008356$. Substituting these values into equation (14), we have:

$$\hat{S}_{pmk} = \frac{\Phi^{-1}\left(\dfrac{1+\dfrac{98}{100}-\dfrac{4}{100}}{2}\right)}{3\sqrt{1+\left(\dfrac{\bar{X}-T}{S}\right)^2}} = \frac{\Phi^{-1}(1-6/200)}{3\sqrt{1+\left(\dfrac{\bar{X}-T}{S}\right)^2}} =$$

$$\frac{1.881}{3\sqrt{1+\left(\dfrac{59.9903-60.000}{0.008356}\right)^2}} = 0.4092.$$

Although $\hat{S}_{pmk}= 0.4092$ is small, indicating that the process is not capable, proportion of nonconformity can be well reflected by computing $\hat{P}'_{Spmk} = 2 \times (1 - \Phi(3\hat{S}_{pmk}\sqrt{1+(\frac{\bar{X}-T}{S})^2}) = 0.06$, which can be verified by noting that there are six observations falling outside the specification limits (59.981, 60.004).

Conclusions

In this paper, we have reviewed $C_p(u, v)$ and $C_{Np}(u, v)$. We have also proposed a new index, S_{pmk}, for non-normal underlying distributions, taking into account process variability, departure from the target value, and proportion of nonconformity. Proportion of nonconformity can be exactly reflected by S_{pmk}. Its superiority over C_{Npmk} has been demonstrated with various non-normal processes. In addition, a method is proposed to estimate S_{pmk} for any underlying distribution. An example illustrating how to calculate the estimate is given. It appears that S_{pmk} possesses practical importance in evaluating process capability. How to improve estimation for S_{pmk} needs to be studied further.

References

Boyles, R.A. (1991), "The Taguchi capability index", *Journal of Quality Technology*, Vol. 23 No. 1, pp. 17-26.

Boyles, R.A. (1994), "Process capability with asymmetric tolerances", *Communications in Statistics – Simulation and Computation*, Vol. 23 No. 3, pp. 615-43.

Chan, L.K., Cheng, S.W. and Spiring, F.A. (1988a), "A new measure of process capability: C_{pm}", *Journal of Quality Technology*, Vol. 20 No. 3, pp. 162-75.

Chan, L.K., Cheng, S.W. and Spiring, F.A. (1988b), "The robustness of the process capability index C_p to departures from normality", in Matusita, K. (Ed.), *Statistical Theory and Data Analysis II*, North Holland, Amsterdam, pp. 223-39.

Chang, P.L. and Lu, K.H. (1994), "PCI calculations for any shape of distribution with percentile", *Quality World*, technical section, September, pp. 110-14.

Chen, J.P. (2000), "Reevaluating the process capability indices for non-normal distributions", *International Journal of Production Research*, Vol. 38 No. 6, pp. 1311-24.

Chen, K.S. and Pearn, W.L. (1997), "An application of non-normal process capability indices", *Quality and Reliability Engineering International*, Vol. 13, pp. 355-60.

Choi, Y.C. and Owen, D.B. (1990), "A study of a new process capability index", *Communications in Statistics – Theory and Methods*, Vol. 19 No. 4, pp. 1231-45.

Chou, Y.M., Owen, D.B. and Borrego, S. (1990), "A. lower confidence limits on process capability indices", *Journal of Quality Technology*, Vol. 22 No. 3, pp. 223-9.

Clements, J.A. (1989), "Process capability calculations for non-normal distributions", *Quality Progress*, September, pp. 95-100.

English, J.R. and Taylor, G.D. (1993), "Process capability analysis – a robustness study", *International Journal of Production Research*, Vol. 31 No. 7, pp. 1621-35.

Gruska, G.F., Mirkhani, K. and Lamberson, L.R. (1989), *Non Normal Data Analysis*, Multiface Publishing Co., Michigan.

Gunter, B.H. (1989a), "The use and abuse of C_{pk}, part 2", *Quality Progress*, Vol. 22 No. 3, pp. 108-9.

Gunter, B.H. (1989b), "The use and abuse of C_{pk}, part 3", *Quality Progress*, Vol. 22 No. 5, pp. 79-80.

Kane, V.E. (1986), "Process capability indices", *Journal of Quality Technology*, Vol. 18 No. 1, pp. 41-52.

Kotz, S., Pearn, W.L. and Johnson, N.L. (1993), "Some process capability indices are more reliable than one might think", *Journal of the Royal Statistical Society, Series C: Applied Statistics*, Vol. 42 No.1, pp. 55-62.

Pearn, W.L. and Chen, K.S. (1995), "Estimating process capability indices for non-normal Pearsonian populations", *Quality and Reliability Engineering International*, Vol. 11 No. 5, pp. 386-8.

Pearn, W.L. and Chen, K.S. (1996), "A Bayesian-like estimator of C_{pk}", *Communications in Statistics – Simulation and Computation*, Vol. 25 No. 2, pp. 321-9.

Pearn, W.L., Kotz, S. and Johnson, N.L. (1992), "Distributional and inferential properties of process capability indices", *Journal of Quality Technology*, Vol. 24 No. 4, pp. 216-33.

SAS Institute, Inc. (1990), *SAS Language: Reference*, Version 6, First Edition. SAS Institute, Inc., Cary, NC.

Schneider, H., Pruett, J. and Lagrange, C. (1995), "Uses of process capability indices in the supplier certification process", *Quality Engineering*, Vol. 8 No. 2, pp. 225-35.

Shapiro, S.S. and Wilk, M.B. (1965), "An analysis of variance test for normality (complete samples)", *Biometrika*, Vol. 52, pp. 591-611.

Somerville, S.E. and Montgomery, D.C. (1997), "Process capability indices and non-normal distributions", *Quality Engineering*, Vol. 9 No. 2, pp. 305-16.

Tong, L.I. and Chen, J.P. (1998), "Lower confidence limits of process capability indices for non-normal process distributions", *International Journal of Quality & Reliability Management*, Vol. 15 No. 8/9, pp. 907-19.

Vännman, K. (1995), "A unified approach to capability indices", *Statistica Sinica*, Vol. 5 No. 2, pp. 805-20.

Zwick, D. (1995), "A hybrid method for fitting distributions to data and its use in computing process capability indices", *Quality Engineering*, Vol. 7 No. 3, pp. 601-13.