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A New Algorithm for Optimally Determining Lot-Sizing Policies for a Deteriorating Item in an Integrated Production-Inventory System

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Abstract—In this study, we focus on optimally determining lot-sizing policies for a deteriorating item among all the partners in a supply chain system with a single-vendor and multiple-buyers so as to minimize the average total costs. We revise Yang and Wee's [1] model using the Fourier series to precisely estimate the vendor's inventory holding costs. Also, we transform our revised model into a more concise version by applying an approximation to the exponential terms in the objective function. In order to solve this problem, we analyze the optimality structure of our revised model and derive several interesting properties. By utilizing our theoretical results, we propose a search algorithm that can efficiently solve the optimal solution. Based on our numerical experiments, we show that the proposed algorithm outperforms the existing solution approach in the literature, especially when the number of buyers is larger in the supply chain system.

Keywords—Integrated system, Search algorithm, Lot-sizing policy, Deterioration.

1. INTRODUCTION

This study aims at optimally coordinating lot-sizing policies for a deteriorating item among all the partners in a supply chain system with a single-vendor and multiple-buyers so as to minimize the average total costs. The vendor (which is a producer) distributes a deteriorating item to the buyers. We assume that the replenishment cycle of each buyer, denoted by T_i, must be an integer-ratio fraction of the replenishment cycle of the vendor (denoted by T). That is, $T_i = T/n_i$ and $n_i \in \{1, 2, 3, 4, \dots\}$ for all i.

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Deterioration occurs for most products in the real world. (We note that deterioration means that a product fails to regularly implement its function.) Ghare and Schrader [2] classified the deteriorating properties of inventory into three categories:

(1) direct spoilage, e.g., vegetable, fruit, and fresh food, etc.;
(2) physical depletion, e.g., gasoline and alcohol, etc.;
(3) deterioration such as radiation changing, negative spoiling, and loss of efficacy in inventory, e.g., electronic components and medicine.

From another point of view, deterioration can also be classified by the time-value or the products' life of inventory. Raafat [3] categorized deterioration by the time-value of inventory.

(1) Utility Constant: Its utility does not change significantly as time passes within its valid usage period, e.g., liquid medicine.
(2) Utility Increasing: Its utility increases as time passes, e.g., some alcoholic drinks.
(3) Utility Decreasing: Its utility decreases as time passes, e.g., vegetables, fruits, and fresh foods, etc.

On the other hand, Nahmias [4] classified deterioration by products' lifetime of inventory.

(1) Fixed Lifetime: Products' lifetime is prespecified and its lifetime is independent of the deteriorated factors; therefore, it is called *time-independent* deterioration. In fact, the utility of these products decreases during its lifetime, and when passing its lifetime, the product will perish completely and become of no value, e.g., milk, inventory in blood bank, and food, etc.
(2) Random Lifetime: There is no specified lifetime for these products. The lifetime for these products is assumed as a random variable, and its probability distribution could be a gamma distribution, Weibull distribution, or an exponential distribution, etc. Products that keep deteriorating in some probability distribution are also the so-called *time-dependent* deteriorating products, e.g., electronic components, chemicals, and medicine, etc.

The scope of this study covers those deteriorating products being classified as utility decreasing (as regards their time-value) and also as random lifetime (as regards their lifetime). Furthermore, we assume the deterioration of inventory to be exponentially distributed.

Since deterioration will incur additional costs for inventory storage, it could distort the decision-making scenario and mislead the decision makers' replenishment strategy if one ignores the deteriorating factor in their inventory models. However, most of the inventory models have considered the deteriorating factor as single-product or single-vendor single-buyer models, for instance, [5–9]. In the literature, the present authors have found very few articles that studied inventory models with multiple deteriorating products or single-vendor multibuyer models. Hwang and Moon [10] presented a production-inventory model that integrates the production planning of two products produced on a single facility and the raw material may be deteriorating over time with a constant rate. Kar *et al.* [11] proposed an inventory model for several continuously deteriorating products, sold from two shops under single management dealing with limitations on investment and total floor-space area. On the other hand, the one-warehouse multiretailer problem is one of the most representative studies in the integrated lot-sizing problems. One may refer to the following papers for further reference, namely, [12–16], etc. We note that these papers do not take into account the deteriorating factor in their mathematical models. Recently, some researchers have been working on the integrated lot-sizing models for a deteriorating item in single-vendor and multiple-buyers production-inventory systems. One may refer to [1,17–21] for reference. These inventory models share some common characteristics with the multiple-product inventory models though there still exist significant differences between them, especially in their solution approaches.

In this study, we focus on solving the inventory control problem presented in Yang and Wee's [1] paper. First, we review the assumptions in Yang and Wee's model as follows. There are totally N buyers in this supply chain system. Customer demand occurs with each buyer at a constant rate.

A holding cost is incurred for each unit of finished product per unit time stored, and a setup cost is charged for each order placed with the vendor and with/by each buyer. The demand rates, holding cost rates, and setup costs are stationary for the vendor and each buyer. The production rate of the production facility is finite, and it is greater than the sum of all the buyer's demands. And, no backlogging is permitted anytime in the system. Finally, the replenishment of orders is assumed to be instantaneous (though this assumption can be relaxed by adding lead times to the orders). Also, we define some notation used in Yang and Wee's model as follows. We denote T as the length of the replenishment cycle. And, $T = T_1 + T_2$, where T_1 and T_2 are the length of production time and the length of nonproduction time in the replenishment cycle, respectively. We let the unit usage of raw materials per finished product be f. We set the ordering cost of raw material as K_m. The set-up cost K_p is incurred each time when the vendor starts one run of production. And, the ordering cost K_b incurs for each buyer as an order is placed. We denote d_i as the demand rate of buyer i and p as the production rate of the production facility at the vendor. We let the holding cost per dollar per unit time for raw material be F_m. And, let F_p and F_b be the holding cost rates of the finished product at the vendor and the buyer, respectively. We denote the unit price of raw material as C_m. Also, C_p and C_b are the unit prices of the finished product for the vendor and the buyer, respectively. And, we denote θ_m and θ as the deterioration rate of the raw material and the finished product, respectively.

The rest of the paper is organized as follows. In order to solve Yang and Wee's [1] inventory control problem, we first present a revised model and conduct full theoretical analysis on the optimality structure of the optimal objective value curve in Section 2. Then, we employ our theoretical results to devise a search algorithm that solves the optimal solution for the single-vendor multibuyers system in Section 3. Next, in the first part of Section 4, we present a numerical example to demonstrate the implementation of the proposed search algorithm. Also, based on our random experiments, we show that our search algorithm outperforms Yang and Wee's heuristic in the second part of Section 4. Finally, we address our concluding remarks in Section 5.

2. THEORETICAL ANALYSIS

In this section, we present a mathematical model that optimally coordinates lot-sizing policies for a deteriorating item among all the partners in a supply chain system with a single vendor and multiple buyers. Also, we conduct theoretical analysis on the mathematical model and present some theoretical results that provide insights into the optimality structure of the mathematical model.

2.1. The Mathematical Model

In the first part of this section, we present a revised version of Yang and Wee's [1] model. In our revised model, we derive new mathematical expressions by the Fourier series to precisely estimate the vendor's inventory holding costs. Then, we transform our revised model into a more concise version by applying an approximation to the exponential terms in the objective function. We note that our revised model will later facilitate our theoretical analysis and algorithm design.

In the following discussion, we focus on deriving accurate expressions for computing the vendor's inventory holding costs in our revised model. As shown in Figure 1, there are two phases, namely, $0 \le t \le T_1$ and $T_1 \le t \le T$, regarding the dynamics of the vendor's finished product. We set $T_2 = T - T_1$ for ease of our presentation later. (We note that Figure 1, on p. 572, in Yang and Wee's [1] paper should be corrected by showing that the vendor's inventory level drops after the buyers' replenishment.) In the first phase, the vendor keeps producing the finished item and the buyers could replenish the finished item at the meanwhile. Then, in the second phase, all the buyers consume the vendor's inventory which accumulates by the end of the first phase. In Yang and Wee's [1] paper, they represent the dynamics of the vendor's finished product by the

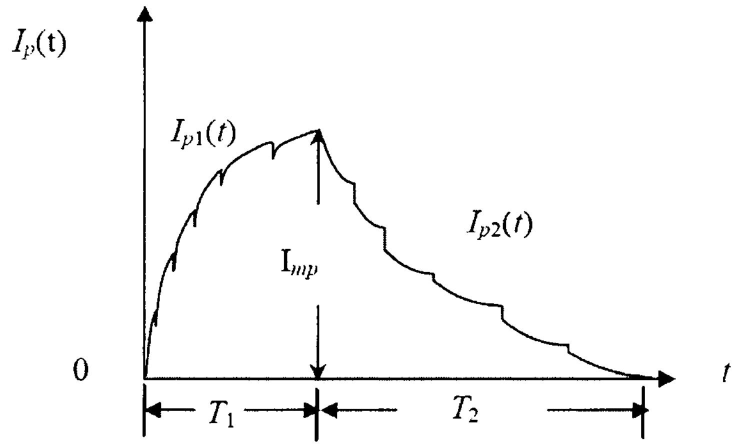

Figure 1. The dynamics of the vendor's finished product.

expressions in (1) and (2) as follows.

$$\frac{dI_{p1}(t)}{dt} + \theta I_{p1}(t) = p - \sum_{i=1}^{N} d_i, \qquad 0 \le t \le T_1, \tag{1}$$

$$\frac{dI_{p2}(t)}{dt} + \theta I_{p2}(t) = - \sum_{i=1}^{N} d_i, \qquad T_1 \le t \le T, \tag{2}$$

where $I_{p1}(t)$ and $I_{p2}(t)$ are the vendor's inventory level of finished product at any time t in the time intervals $[0, T_1]$ and $[T_1, T]$, respectively.

Expressions (1) and (2) indicate that the consumption rate of the vendor's finished product is equal to the sum of the deterioration rate of the vendor's inventory and the summation of all the buyer's demand rates, i.e., $\sum_{i=1}^{N} d_i$. Also, the term p shows in the right side of (1) since the vendor's inventory accumulates due to the production in the first phase.

We would like to point out that the consumption rate of the vendor's inventory in (1) and (2) is *not accurate* since the consumption rate from all the buyers includes not only the summation of all the buyer's demand rates, i.e., $\sum_{i=1}^{N} d_i$, but also the deterioration rate of all the buyers' inventory. In other words, the vendor needs to supply the buyers for both their demands and their consumption from deterioration. A shortage problem may exist if the vendor employs (1) and (2) to plan for its replenishment policy.

Therefore, we propose to use the expressions in (3) and (4) to accurately compute the vendor's finished product hold during the replenishment cycle.

$$\frac{dI_{p1}(t)}{dt} + \theta I_{p1}(t) = p + \sum_{i=1}^{N} \frac{dI_{bi}(t)}{dt}, \qquad 0 \le t \le T_1, \tag{3}$$

$$\frac{dI_{p2}(t)}{dt} + \theta I_{p2}(t) = \sum_{i=1}^{N} \frac{dI_{bi}(t)}{dt}, \qquad T_1 \le t \le T. \tag{4}$$

The dynamics of the vendor's raw material, as illustrated in Figure 2, can be described by the following equation.

$$\frac{dI_m(t)}{dt} + \theta_m I_m(t) = -fp, \qquad 0 \le t \le \frac{T_1}{n_m}, \tag{5}$$

where n_m is the number of deliveries of the raw material from the supplier to the vendor in T_1.

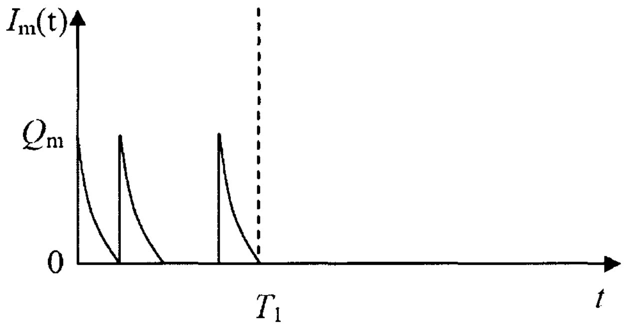

Figure 2. The dynamics of the vendor's raw material.

From the ordinary differential equation in (5) and the boundary condition $I_m(T_1/n_m) = 0$, we may express the dynamics of the vendor's raw material as

$$I_m(t) = \frac{fp}{\theta_m}\left(\frac{e^{\theta_m T_1/n_m} - e^{\theta_m t}}{e^{\theta_m t}}\right) = \frac{fp}{\theta_m}\left[e^{\theta_m T_1/n_m} e^{-\theta_m t} - 1\right], \qquad 0 \le t \le \frac{T_1}{n_m}. \tag{6}$$

On the other hand, the dynamics of the finished product of buyer i in the time period $[0, T/n_i]$ can be expressed as follows.

$$s\frac{dI_{bi}(t)}{dt} + \theta I_{bi}(t) = -d_i, \qquad 0 \le t \le \frac{T}{n_i}. \tag{7}$$

One may observe that the consumption rate of the finished product of buyer i includes not only the demand rate from its customer, but also from deterioration. By solving the differential equations in (7), we have

$$I_{bi}(t) = \frac{d_i}{\theta}\left(\frac{\left(e^{\theta T/n_i} - e^{\theta t}\right)}{e^{\theta t}}\right) = \frac{d_i}{\theta}\left[e^{\theta T/n_i} e^{-\theta t} - 1\right], \qquad 0 \le t \le \frac{T}{n_i}. \tag{8}$$

Hence, we have the closed form for the consumption rate of the finished product of buyer i given by

$$\frac{dI_{bi}(t)}{dt} = -d_i e^{\theta(T/n_i)} e^{-\theta t}. \tag{9}$$

During the replenishment cycle T, there should be n_i times of replenishment to buyer i. Therefore, the dynamics of each individual buyer's finished product shall repeat periodically. We note that it is important to obtain an accurate expression for $\sum_{i=1}^{N} \frac{dI_{bi}(t)}{dt}$ so as to precisely estimate the vendor's holding costs for the finished product as shown in equations (3) and (4). However, since the replenishment cycle of each buyer may not be the same, it is hard to *directly* calculate the sum of the consumption rates of the finished product of all the buyers (which is used in the right-side equations (3) and (4)). One may refer to Figure 3 for an example of three buyers with different replenishment cycles.

French mathematician J. Fourier commented that any periodic motion can be represented by a series of *sines* and *cosines* which are harmonically related. (One may refer to [22] for reference.) Using the Fourier series, we can represent the exponential term in equation (9) by

$$e^{-\theta t} = \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n \cos \omega_n t + b_n \sin \omega_n t), \tag{10}$$

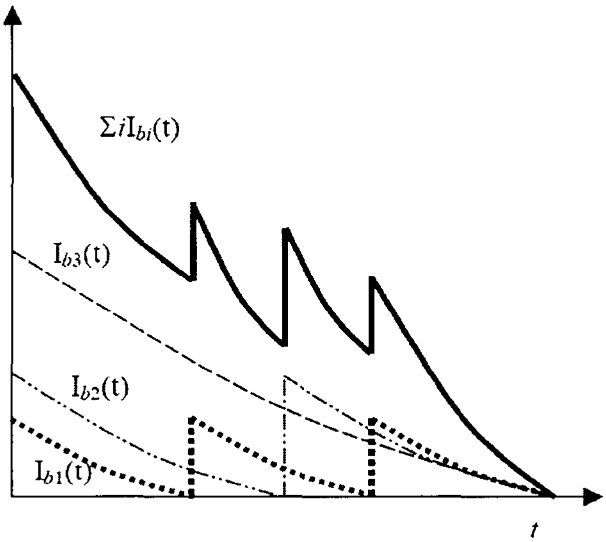

Figure 3. The dynamics of the sum of the finished product of three buyers.

where

$$a_0 = \frac{1}{(T/2n_i)\theta}\left(e^{\theta(T/2n_i)} - e^{-\theta(T/2n_i)}\right), \qquad \omega_n = \frac{2n\pi}{T/n_i},$$

$$a_n = \frac{1}{(T/2n_i)}\int_{-T/2n_i}^{T/2n_i} e^{-\theta t}\cos\omega_n t\,dt = \frac{(-1)^n\theta}{(T/2n_i)\left(\theta^2+\omega_n^2\right)}\left(e^{\theta(T/2n_i)} - e^{-\theta(T/2n_i)}\right), \qquad \text{and}$$

$$b_n = \frac{1}{(T/2n_i)}\int_{-T/2n_i}^{T/2n_i} e^{-\theta t}\sin\omega_n t\,dt = \frac{(-1)^n\omega_n}{(T/2n_i)\left(\theta^2+\omega_n^2\right)}\left(e^{\theta(T/2n_i)} - e^{-\theta(T/2n_i)}\right).$$

Recall that there are two phases, namely, $0 \le t \le T_1$ and $T_1 \le t \le T$, in the replenishment cycle T. By using the Fourier series and the boundary condition $I_{p1}(0) = 0$, we may derive the dynamics of the vendor's finished product in the first phase by (11) as follows.

$$I_{p1}(t) = \left(1 - e^{-\theta t}\right)\frac{p - \sum\limits_{i=1}^{N} d_i e^{\theta T/n_i}}{\theta} - e^{-\theta t}\sum_{i=1}^{N} d_i e^{\theta T/n_i}\left\{\sum_{n=1}^{\infty} a_n h_1(t) + \sum_{n=1}^{\infty} b_n h_2(t)\right\}, \qquad (11)$$
$$0 \le t \le T_1,$$

where

$$h_1(t) = \frac{1}{\theta^2+\omega_n^2}\left[\left(\theta\cos\omega_n t + \omega_n\sin\omega_n t\right)e^{\theta t} - \theta\right)\right]$$

and

$$h_2(t) = \frac{1}{\theta^2+\omega_n^2}\left[\left(\theta\sin\omega_n t - \omega_n\cos\omega_n t\right)e^{\theta t} + \omega_n\right)\right].$$

The terms with a_n and b_n in (11) come from the Fourier series expansion, and they assist to match with the saw-toothed curve for the dynamics of the vendor's finished product as shown in Figure 1.

Similarly, we may represent the dynamics of the vendor's finished product in the second phase as

$$\begin{aligned}
I_{p2}(t) &= e^{-\theta t}\left\{\int e^{\theta t'}\sum_{i=1}^{N} A_i\left[\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\omega_n t'+b_n\sin\omega_n t'\right)\right]dt'+c_2\right\} \\
&= \frac{\sum_{i=1}^{N} d_i}{\theta}e^{\theta T/n_i}\left(e^{\theta T}e^{-\theta t}-1\right)+e^{-\theta t}\sum_{i=1}^{N} d_i e^{\theta T/n_i}\left\{\sum_{n=1}^{\infty} a_n g_1(t)+\sum_{n=1}^{\infty} b_n g_2(t)\right\}+c_2, \qquad (12)\\
&T_1 \le t \le T,
\end{aligned}$$

where

$$g_1(t) = \frac{1}{\theta^2+\omega_n^2}\left[(\theta\cos\omega_n t+\omega_n\sin\omega_n t)e^{\theta t}-(\theta\cos\omega_n T_1-\omega_n\sin\omega_n T_1)e^{\theta T_1}\right],$$

$$g_2(t) = \frac{1}{\theta^2+\omega_n^2}\left[(\theta\sin\omega_n t-\omega_n\cos\omega_n t)e^{\theta t}-(\theta\sin\omega_n T_1-\omega_n\cos\omega_n T_1)e^{\theta T_1}\right],$$

and the coefficient c_2 can be obtained by using the boundary condition $I_{p2}(T)=0$.

Now, we are ready to compute the average inventory level of the vendor's finished product by

$$\frac{1}{T}\int_0^T I_p(t)\,dt = \frac{1}{T}\int_0^{T_1} I_{p1}(t_1)\,dt_1+\frac{1}{T}\int_{T_1}^{T} I_{p2}(t_2)\,dt_2. \qquad (13)$$

We note that our revised model is a notorious nonlinear-integer program, and it also carries many exponential terms. Its formulation not only makes the theoretical analysis on our revised model extremely difficult, but also gives rise to the difficulty of solving it directly. Therefore, we suggest the use of an approximation in (14), which is proposed by Yao and Wang [23], to simplify the term $e^{\theta t}$ in its formulation.

$$e^{\theta t}\approx 1+\theta t+\frac{\theta^2t^2}{2}+\frac{\theta^3t^2}{3!}, \qquad 0<\theta t<1. \qquad (14)$$

We derive our approximation from the Taylor series expansion. Also, Yao and Wang [23] show that the approximation in equation (14) achieves better precision in approximating the term $e^{\theta t}$ than another function in (15) which is popularly used in the literature, namely,

$$e^{\theta t}\cong\frac{(2+\theta t)}{(2-\theta t)}. \qquad (15)$$

(One may refer to [24] for details.)

After applying the approximation in equation (14), one may have a more friendly formulation for each cost term in our revised model as follows: after using our approximation, the annual holding cost for the vendor's finished product is given by

$$\begin{aligned}
\mathrm{HC}_p &= \frac{C_pF_p}{T}\int_0^T I_p(t)\,dt \\
&= \frac{C_pF_p}{T}\left\{\frac{p}{\theta}\left(T_1+\frac{e^{-\theta T_1}-1}{\theta}\right)-\frac{a_0}{2\theta}\sum_{i=1}^{N} d_i e^{\theta(T/n_i)}\left[T-T_1+\frac{1}{\theta}\left(1-e^{\theta T}e^{-\theta T_1}\right)\right]\right\} \qquad (16)\\
&\approx \frac{C_pF_p}{T}\left\{\frac{p}{2}T_1^2-\frac{a_0}{4}\sum_{i=1}^{N} d_i e^{\theta(T/n_i)}\left(T-T_1\right)^2\right\},
\end{aligned}$$

where the term $e^{\theta(T/n_i)}$ in (16) could be further approximated using the expression in (13). We may express the variable T_1 in (16) in terms of T by using the boundary condition $I_{p1}(T_1) = I_{p2}(T_1)$ as follows.

$$T_1 \approx \left(T + \frac{\theta}{n_i}T^2\right)\frac{\sum_{i=1}^{N} d_i}{p}. \tag{17}$$

On the other hand, the annual holding cost for all the buyers' finished product can be expressed as follows.

$$\mathrm{HC}_b = \frac{C_b F_b}{T}\sum_{i=1}^{N} n_i \int_0^{T/n_i} I_{bi}(t)\,dt \approx \frac{C_b F_b}{6}\sum_{i=1}^{N} d_i\left(3 + \frac{\theta}{n_i}\right)\frac{T}{n_i}. \tag{18}$$

The annual holding cost of the vendor's raw material is expressed as

$$\mathrm{HC}_m = \frac{C_m F_m n_m}{T}\int_0^{T_1/n_m} I_m(t)\,dt = \frac{C_m F_m f p}{6}\left(3 + \frac{\theta}{n_m}\right)\frac{T}{n_m}. \tag{19}$$

If we denote DC_b and DC_p as the annual deterioration costs for all the buyers and the vendor, respectively, then we have the approximation terms for them as follows.

$$\mathrm{DC}_b = \frac{C_b}{T}\sum_{i=1}^{N} n_i\left(I_{mi} - \frac{d_i T}{n_i}\right) \approx \frac{C_b\theta}{6}\sum_{i=1}^{N} d_i\left(3 + \frac{\theta}{n_i}\right)\frac{T}{n_i} \tag{20}$$

and

$$\mathrm{DC}_p = \frac{C_p}{T}\left(pT_1 - \sum_{i=1}^{N} n_i I_{mi}\right) \approx C_p\left[\frac{pT_1}{T} - \sum_{i=1}^{N} d_i\left(1 + \frac{\theta T}{2n_i} + \frac{\theta^2 T}{6n_i^2}\right)\right]. \tag{21}$$

The annual deteriorated costs for the raw materials is

$$\mathrm{DC}_m = \frac{C_m}{T_1}n_m\left(Q_m - \frac{fpT_1}{n_m}\right) = \frac{C_m f p \theta_m T_1}{2n_m} \approx \sum_{i=1}^{N} d_i\left(\frac{C_m f \theta_m}{2n_m}\right)T. \tag{22}$$

The annual ordering costs for the vendor and for all the buyers are given by (23) and (24), respectively.

$$\mathrm{SC}_m = \frac{K_m n_m}{T_1}, \tag{23}$$

$$\mathrm{SC}_b = \sum_{i=1}^{N}\frac{n_i K_b}{T}. \tag{24}$$

And, we denote SC_p as the setup costs per year for the vendor. Then,

$$\mathrm{SC}_p = \frac{K_p}{T}. \tag{25}$$

The average total costs for all the buyers are the sum of HC_b, DC_b, and SC_b which are expressed in (18), (20), and (24), respectively. On the other hand, the average total costs for the vendor are the sum of HC_p, DC_p, and SC_p which are expressed in (16), (21), and (25), respectively. The average total costs for the raw material are the sum of HC_m, DC_m, and SC_m which are expressed in (19), (22), and (23), respectively. Therefore, the annual total costs of the integrated system are given by

$$\mathrm{TC} = \mathrm{HC}_p + \mathrm{DC}_p + \mathrm{SC}_p + \mathrm{HC}_b + \mathrm{DC}_b + \mathrm{SC}_b + \mathrm{HC}_m + \mathrm{DC}_m + \mathrm{SC}_m,$$

which can be rewritten as follows.

$$TC = \frac{K_p}{T} + \frac{K_m n_m p \Big/ \sum_{i=1}^{N} d_i}{T} + \sum_{i=1}^{N} \left[d_i T \left(C_1 + \frac{C_2}{n_i} \right) + \frac{n_i K_b}{T} \right], \tag{26}$$

where $C_1 = C_p F_p/2$ and $C_2 = (C_p\theta + C_b F_b + C_b\theta)/2$. We note that C_1 and C_2 are constant terms.

Let $f_i = 1/n_i$. Then, we may formulate a mathematical model, namely, problem (P) for optimally coordinating lot-sizing policies of a deteriorating item in a supply chain system with a single-vendor and multiple-buyers as follows.

(P)

$$\text{Minimize} \quad \mathrm{TC}\,(f_1, \dots, f_N, T, n_m) \equiv \frac{K_p}{T} + \frac{K_m n_m p \Big/ \sum_{i=1}^{N} d_i}{T} + \sum_{i=1}^{N} \left[d_i C_1 T + TC_i(f_i, T) \right], \tag{27}$$

$$\text{subject to} \quad f_i \in \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}, \qquad i = 1, \dots, N, \tag{28}$$

$$\text{where} \quad \mathrm{TC}_i(f_i, T) = \frac{K_b}{f_i T} + d_i C_2 T f_i. \tag{29}$$

Interestingly, we may note that problem (P) is *separable* in terms of the decision variables $\{f_i : i = 1, \dots, N\}$.

2.2. Some Insights into the Optimal Cost Function

In this section, we analyze the function $\mathrm{TC}_i(f_i, T)$. For a given $T = T'$, one can solve the optimal multiplier f_i so as to minimize the value of $\mathrm{TC}_i(f_i, T = T')$. We define it as $\underline{\mathrm{TC}}_i(T)$, the minimum cost function for buyer i with respect to all the value of T' on the T-axis, i.e.,

$$\underline{\mathrm{TC}}_i(T) = \bigcup_{T' \in R^+} \min_{f_i \in p^{-1},\ \text{where}\ p \in N^+} \left\{ \mathrm{TC}_i\left(f_i, T'\right) \right\}. \tag{30}$$

Then, Lemma 1 holds for each buyer i.

LEMMA 1. *$\underline{\mathrm{TC}}_i(T)$ is a piece-wise convex function with respect to T. Also, for each value of f_i, one can obtain the local minima for $\underline{\mathrm{TC}}_i(T)$ at*

$$\lambda_i\left(f_i\right) = \frac{1}{f_i} \sqrt{\frac{K_b}{d_i C_2}} \tag{31}$$

with the minimum cost of $\underline{\mathrm{TC}}_i(\lambda_i(f_i)) = 2\sqrt{K_b d_i C_2}$.

PROOF. It can be shown by an easy algebraic derivation. ∎

One may refer to the graphical representation of Lemma 1 in Figure 4.

Similarly, we define $\mathrm{TC}_{\mathrm{opt}}(T)$ as the optimal objective function value of problem (P) with respect to T, i.e.,

$$\mathrm{TC}_{\mathrm{opt}}(T) \equiv \frac{K_p}{T} + \frac{K_m n_m p \Big/ \sum_{i=1}^{N} d_i}{T} + \sum_{i=1}^{N} \left[d_i C_1 T + \underline{\mathrm{TC}}_i(T) \right]. \tag{32}$$

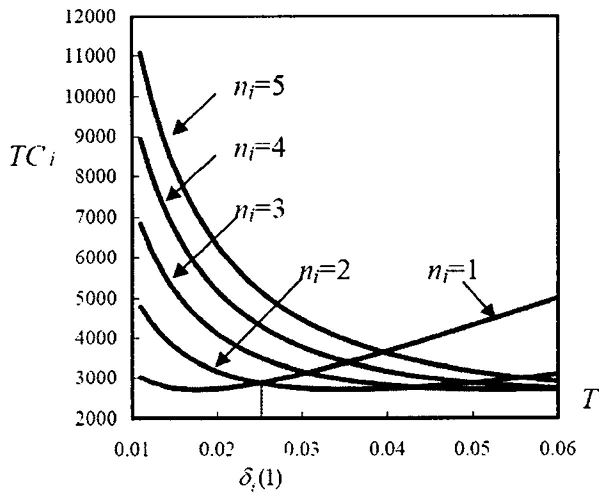

Figure 4. The piece-wise convex curve of the $\underline{\mathrm{TC}}_i(T)$ function.

The following proposition depicts an important characteristic of the optimal objective function value of problem (P).

PROPOSITION 1. *The* $\mathrm{TC}_{\mathrm{opt}}(T)$ *function is piece-wise convex with respect to* T.

PROOF. It is obvious that $K_p/T + (K_m n_m p/\sum_{i=1}^{N} d_i)/T$ and $d_i C_1 T$ are *convex* functions with respect to T. On the other hand, each $\underline{\mathrm{TC}}_i(T)$ function is *piece-wise convex* with respect to T by Lemma 1. Since $\mathrm{TC}_{\mathrm{opt}}(T)$ is the sum of $(N+1)$ convex functions and N *piece-wise* convex functions by (32), it is surely a *piece-wise* convex function. ∎

By utilizing our theoretical results on $\underline{\mathrm{TC}}_i(T)$, we could have more insights into the optimality structure of problem (P).

2.3. The Junction Points

Next, we introduce the "junction points" on the curve of the $\mathrm{TC}_{\mathrm{opt}}(T)$ function. We define a junction point for the $\underline{\mathrm{TC}}_i(T)$ function as a particular value of T where two convex curves using consecutive integers of n_i concatenate. (For example, as shown in Figure 1, the junction point $\delta_i(1)$ is the particular value of T where the two consecutive convex curves, with $n_i = 1$ and $n_i = 2$, concatenate.) These junction points determine at '*what value of* T' where one should change the multiplier of buyer i from f_i to $f_i/(f_i+1)$ so as to secure the minimum value for the $\underline{\mathrm{TC}}_i(T)$ function. We first derive a closed-form for the location of the junction points for buyer i as follows. We define the difference function $\Delta_i(f_i, T)$ by

$$\Delta_i(f_i, T) = TC_i\left(\frac{f_i}{f_i+1}, T\right) - TC_i(f_i, T) = \frac{K_b}{T} - d_i C_2 T \frac{f_i^2}{f_i+1}. \tag{33}$$

We note that $\Delta_i(f_i, T)$ is the cost difference between using f_i and $f_i/(f_i+1)$ as its multiplier for $\underline{\mathrm{TC}}_i(f_i, T)$. Since the function $\Delta_i(f_i, T)$ is an increasing function with respect to T, suppose that the search algorithm proceeds from a lower bound toward larger values of T; we evaluate $\Delta_i(f_i, T)$ from positive values, to zero, and finally, to negative values. Let ω be the point where $\Delta_i(f_i, T)$ reaches zero. Assume that f_i is the optimal multiplier for buyer i for $T < \omega$. This scheme implies that one should keep using f_i until it meets ω. From the point ω onwards, the value of $\underline{\mathrm{TC}}_i(f_i, T)$ can be improved by using $f_i/(f_i+1)$ as its optimal multiplier. We note that ω is the point where two neighboring convex curves $\mathrm{TC}_i(f_i, T)$ and $\mathrm{TC}_i(f_i/(f_i+1), T)$ meet. Importantly, such a junction point ω provides us with the information not only on "*which buyer i*" to modify, but also on "*where on the T-axis*" to replace f_i with $f_i/(f_i+1)$. By equation (33), we identify a junction point for buyer i by

$$\delta_i\left(\frac{1}{f_i}\right) = \sqrt{\frac{f_i+1}{f_i^2}}\sqrt{\frac{K_b}{d_i C_2}}. \tag{34}$$

More specifically, $\delta_i(1/f_i = 1/j)$ is the $(1/j)^{\text{th}}$ junction point of buyer i where $(1/j) \in N^+$. Therefore, the junction point $\delta_i(1/j)$ provides us with the information that one should choose $f_i = j$ for $T < \delta_i(1/j)$ and choose $f_i = j/(j+1)$, *vice versa*, to obtain the lowest value for the $\underline{\text{TC}}_i(T)$ function.

The following theoretical results on the junction points provide a strengthened foundation for such a search scheme.

LEMMA 2. *Suppose that $f_i^{(L)}$ and $f_i^{(R)}$, respectively, are the optimal multipliers of for the convex curves on the left side and the right side of a junction point of the $\underline{\text{TC}}_i(T)$ function. Then,*

$$f_i^{(R)} = \frac{f_i^{(L)}}{f_i^{(L)}+1}. \tag{35}$$

PROOF. It can be easily proven by (34). ∎

PROPOSITION 2. *All the junction points for each individual buyer i, will be inherited by the $\text{TC}_{\text{opt}}(T)$ curve.*

PROOF. Recall that $\text{TC}_{\text{opt}}(T) \equiv K_p/T + (K_m n_m p/\sum_{i=1}^N d_i)/T + \sum_{i=1}^N [d_i C_1 T + TC_i(f_i, T)]$ is separable. Assume that ω is a junction point for buyer i, but not for the other $(i-1)$ buyers. Then, there must exist $\varepsilon > 0$ such that the following two facts hold.

1. The curve for $K_p/T + (K_m n_m p/\sum_{i=1}^N d_i)/T + \sum_{i=1}^N d_i C_1 T + \sum_{j\neq i} \underline{\text{TC}}_j(T)$ is convex in the interval of $[\omega-\varepsilon, \omega+\varepsilon]$ since each one of $\underline{\text{TC}}_j(T)$ where $j \neq i$ is convex in $[\omega-\varepsilon, \omega+\varepsilon]$, and
2. $\underline{\text{TC}}_i(T)$ is convex in the intervals of $[\omega, \omega+\varepsilon]$. Except at the junction point ω, $\text{TC}_{\text{opt}}(T) = K_p/T + (K_m n_m p/\sum_{i=1}^N d_i)/T + \sum_{i=1}^N d_i C_1 T + \sum_{j\neq i} \underline{\text{TC}}_j(T) + \sum_i \underline{\text{TC}}_i(T)$ is still convex in the intervals $[\omega-\varepsilon, \omega]$ and $[\omega, \omega+\varepsilon]$.

Therefore, ω becomes a junction point of $\text{TC}_{\text{opt}}(T)$. ∎

In other words, Proposition 2 asserts that if a junction point ω shows on one piece-wise convex curve $\underline{\text{TC}}_i(T)$, then, ω must also show on the piece-wise convex curve of the $\text{TC}_{\text{opt}}(T)$ function as a junction point. Define $f_i^*(T)$ as the optimal multiplier for buyer i given a particular value of $T \in R^+$. Let $F(T)$ be the vector of optimal multipliers at a given T, i.e., $F(T) = (f_1^*(T), \dots, f_N^*(T))$. The following theorem is an immediate result of Lemma 2 and Proposition 2.

THEOREM 1. *Suppose that ω is a junction point in the plot of the $\text{TC}_{\text{opt}}(T)$ function. Also, $F^{(L)} = (f_1^{(L)}(\omega), f_2^{(L)}(\omega), \dots, f_N^{(L)}(\omega)) \equiv F(\omega-\varepsilon) = (f_1^*(\omega-\varepsilon), f_2^*(\omega-\varepsilon), \dots, f_N^*(\omega-\varepsilon))$ and $F^{(R)} = (f_1^{(R)}(\omega), f_2^{(R)}(\omega), \dots, f_N^{(R)}(\omega)) \equiv F(\omega+\varepsilon) = (f_1^*(\omega+\varepsilon), f_2^*(\omega+\varepsilon), \dots, f_N^*(\omega+\varepsilon))$ are the vectors of the optimal multipliers for the left-side and right-side convex curves with regard to the junction point ω, respectively. Then, $F^{(R)}$ is secured from $F^{(L)}$ by changing at least one of its optimal multiplier by $f_n^{(R)}(\omega) = f_n^{(L)}(\omega)/(f_n^{(L)}(\omega)+1)$.*

Usually, only one f_i^* changes at a junction point except for some extreme cases in which two buyers share the same junction point.

The following corollary is also a by-product of Lemma 2 and Proposition 2, and it provides an easier way to obtain each $f_i^*(T) \in F(T)$.

COROLLARY 1. *For any given T, one can obtain each $f_i^*(T) \in F(T)$ by*

$$f_i^*(T) = \begin{cases} 1, & T < \sqrt{\dfrac{2K_b}{d_i C_2}}, \\ m, & \sqrt{\dfrac{1-m}{m^2}}\sqrt{\dfrac{K_b}{d_i C_2}} < T \leq \sqrt{\dfrac{1+m}{m^2}}\sqrt{\dfrac{K_b}{d_i C_2}}. \end{cases} \tag{36}$$

The following corollary is important for the design of the proposed search algorithm.

COROLLARY 2. *Let ω_1 and ω_2 be two neighboring junction points for the function $\mathrm{TC}_{\mathrm{opt}}(T)$, and $\omega_1 < \omega_2$. Then, the vector of optimal multipliers for the $\mathrm{TC}_{\mathrm{opt}}(T)$ function is invariant in (ω_1, ω_2).*

PROOF. It is obvious from Theorem 1, where we know that $F(\omega_2)$ is obtained from $F(\omega_1)$ by changing at least one of its optimal multipliers by $f_i^*(\omega_2) = f_i^*(\omega_1)/(f_i^*(\omega_1) + 1)$. Thus, the vector of optimal multipliers for the $\mathrm{TC}_{\mathrm{opt}}(T)$ function is invariant in (ω_1, ω_2). ∎

3. THE PROPOSED SEARCH ALGORITHM

In this section, we propose a search algorithm that obtains a heuristic solution for problem (P).

First, we provide an overview of the proposed search algorithm before presenting the details. We divide the decision variables in problem (P) into three categories, namely, n_m, T, and $(n_1, \dots, n_N)$ (or, equivalently, $(f_1, \dots, f_N)$). To make our presentation more concise, we define $F^* \equiv (f_1, \dots, f_N)$ as the vector of the optimal multipliers. The proposed search algorithm starts with setting $n_m = 1$, and search for the optimal solution of problem (P) given $n_m = 1$, namely, $T^*|_{n_m=1}$ and $F^*|_{n_m=1}$. We record the best-on-hand solution by setting $\mathrm{TC}^{\mathrm{BOH}} = \mathrm{TC}(F^*|_{n_m=1}, T^*|_{n_m=1}, n_m = 1)$, $T^{\mathrm{BOH}} = T^*|_{n_m=1}$, and $F^{\mathrm{BOH}} = F^*|_{n_m=1}$. Then, we increase the value of n_m by 1, i.e., $n_m = n_m + 1 = 2$, and obtain the optimal solution given $n_m = 2$, namely, $T^*|_{n_m=2}$ and $F^*|_{n_m=2}$. Next, if the latest-obtained optimal solution is better than the best-on-hand solution, we update it by setting $\mathrm{TC}^{\mathrm{BOH}} = \mathrm{TC}(F^*|_{n_m=2}, T^*|_{n_m=2}, n_m = 2)$, $T^{\mathrm{BOH}} = T^*|_{n_m=2}$, and $F^{\mathrm{BOH}} = F^*|_{n_m=2}$; otherwise, we terminate the search algorithm. That is, we repeat such a search scheme until we are not able to improve the best-on-hand solution.

We design the search scheme discussed above based on our observations on extensive numerical experiments. We have strong confidence that such a search scheme may solve the optimal solution for problem (P). However, we could merely call it a *heuristic* since it is extremely difficult to provide a rigorous proof for the characteristic that the envelop of the optimal objective function value with respect to n_m is convex. (Note: our termination condition is devised based on such a convexity characteristic, and one may refer to Figure 5 in Section 4.1 for its graphical representation. We will discuss it later that solving problem (P) given a particular value of n_m is not trivial at all. Our discussions in Sections 3.1–3.3 might help to learn the difficulty to prove the convexity characteristic mentioned above.)

Next, we propose a search algorithm that could obtain the optimal solution for problem (P) given a particular value of n_m. Recall that the $\mathrm{TC}_{\mathrm{opt}}(T)$ function is piece-wise convex with respect to T. (One may refer to Section 2.) Also, some interesting properties on the junction points reveal the optimality structure of the $\mathrm{TC}_{\mathrm{opt}}(T)$ function. These theoretical results encourage us to solve problem (P) by searching along the T-axis.

To design such a search algorithm, we first need to define the search range by a lower and an upper bound on the T-axis, which are denoted by T_L and T_U, respectively. We note that the bounds T_L and T_U are derived by asserting that the best local minimum in $[T_L, T_U]$ must be no worse than any solution outside of $[T_L, T_U]$. Naively, one can solve an optimal solution by a small-step search algorithm which enumerates $T \in [T_L, T_U]$ and using a very small step-size $\Delta T \to 0$. But, this is neither efficient nor accurate, since the step-size determines its performance. Also, the run time of the search algorithm may be extremely long if the search range $[T_L, T_U]$ is wide.

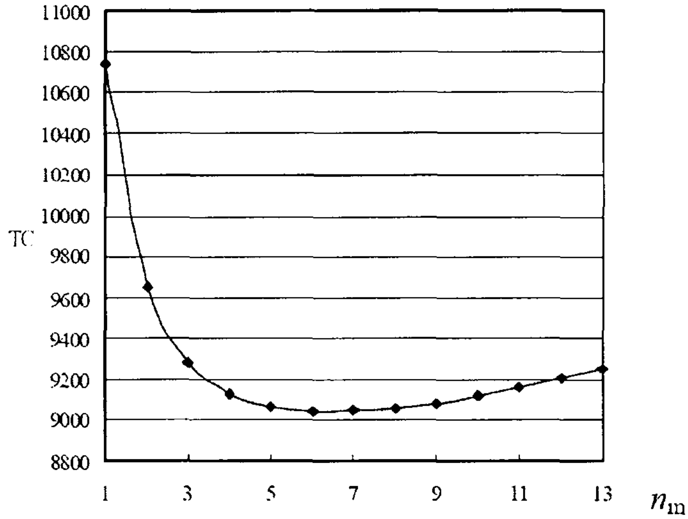

Figure 5. The optimal objective function value of problem (P) vs. different values of n_m.

In order to propose an efficient search algorithm, we must utilize our theoretical results on the optimality structure, especially the properties of the junction points on the $\mathrm{TC}_{\mathrm{opt}}(T)$ function. By Lemma 2 and Proposition 2, we can easily obtain all of the junction points within any search range $[T_L, T_U]$ by equation (34). Corollary 2 asserts that the vector of optimal multipliers for $\mathrm{TC}_{\mathrm{opt}}(T)$ is invariant in any convex interval between two neighboring junction points. These theoretical results lead us to the following idea: if we are able to obtain all of the local minima for each convex curve in $[T_L, T_U]$, we surely can obtain an optimal solution by picking the one with the lowest objective value.

In the following sections, we first derive a lower bound on the search range in Section 3.1. Then, Section 3.2 demonstrates how to use the junction points to proceed with the search. Also, we propose an approach to secure and revise the upper bound on the search range in Section 3.3. (Note: our discussions in Sections 3.1–3.3 provide the details of our search algorithm to solve problem (P) given a particular value of n_m. Therefore, we treat n_m as a constant in these three sections.) Finally, we summarize our proposed search algorithm.

3.1. A Lower Bound

In this section, we derive a lower bound on the search range by the *common cycle* (CC) approach which requires that $f_i = 1$ for all i, i.e., all the retailers share the same replenishment cycle.

Denoting as T^*_{cc}, the optimal replenishment cycle for the CC approach, then, one may easily secure T^*_{cc} by the following expression.

$$T^*_{\mathrm{cc}} = \sqrt{\frac{\left(K_p + K_m n_m p \Big/ \sum_{i=1}^{N} d_i + N K_b\right)}{\sum_{i=1}^{N} d_i (C_1 + C_2)}}. \tag{37}$$

Proposition 3 asserts that the search scheme may skip the range $(0, T^*_{\mathrm{cc}})$. Consequently, we may set T^*_{cc} in equation (37) as a lower bound of the search range.

PROPOSITION 3. *For the* $\mathrm{TC}_{\mathrm{opt}}(T)$ *function, there exist no local minima for* $T < T^*_{\mathrm{cc}}$.

PROOF. Proposition 1 asserts that $\mathrm{TC}_{\mathrm{opt}}(T)$ function is *piece-wise convex*. It implies that the optimal solution must be one of its local minima. The local minimum for any vector of $(f_1^*, \dots, f_N^*)$, where $1/f_i^* \in N^+$, $\forall i$, is given by

$$\breve{T}(f_1^*, \dots, f_N^*) = \sqrt{\frac{\left(K_p + K_m n_m p \Big/ \sum_{i=1}^{N} d_i + \sum_{i=1}^{N} (K_b/f_i^*)\right)}{\sum_{i=1}^{N} d_n \left[C_1 + C_2 f_i^*\right]}}. \tag{38}$$

By equation (38), it is obvious that $\breve{T}(f_1^*, \dots, f_N^*) \geq T_{\mathrm{cc}}^*$ since $f_i^* \leq 1$ for all i. Therefore, there exists no local minima for $T < T_{\mathrm{cc}}^*$. ∎

3.2. Proceeding with the Search by Junction Points

By utilizing the theoretical properties of the junction points, we show how to proceed with the search from our lower bound T_L in this section.

Before proceeding with the search, we first secure $F(T_L)$, i.e., the vector of optimal multipliers at T_L by Corollary 1.

Next, we show how to proceed with the search by utilizing a sequence of (sorted) junction points. By Lemma 2 and Proposition 2, each junction point $\{\delta_i(1/f_i^*)\}$ provides the information that one should change the optimal multiplier of buyer i from f_i^* to $f_i^*(f_i^* + 1)$ at $\delta_i(1/f_i^*)$ to secure the optimal value for the $\mathrm{TC}_{\mathrm{opt}}(T)$ function. Therefore, during the search, we need to keep an *n-dimensional* array $(\delta_1(1/f_1^*), \dots, \delta_N(1/f_N^*))$ in which each value of $\delta_i(1/f_i^*)$ indicates the location of the *next* junction point of each buyer i where the optimal multiplier of buyer i should be changed. Since the algorithm searches toward larger values of T, one shall change the multiplier for the particular buyer i with the *smallest* value of $\delta_i(1/f_i^*)$ to correctly update the vector of optimal multipliers. Let T_c be the current value of T where the search algorithm is reached. Denote as π the index for the buyer i with the *smallest* value of $\delta_i(1/f_i^*)$, i.e., $\pi = \arg\min_i\{\delta_i(1/f_i^*) > T_c\}$. To proceed with the search form T_c, by Theorem 1, we need to update the vector of optimal multipliers at $\delta_i(1/f_i^*)$ by

$$F\left(\delta_\pi\left(\frac{1}{f_\pi^*}\right)\right) \equiv \left(F(T_c) \setminus \{f_\pi^*\}\right) \cup \left\{\frac{f_\pi^*}{(f_\pi^* + 1)}\right\}, \tag{39}$$

where '\' denotes set subtraction.

Let $\{\omega_j\}$ be the sequence of the points where the search algorithm is reached. Also, by the definition, we have $\omega_0 \equiv T_L$. From another point of view, the algorithm searches along $\{\omega_j\}$, a (sorted) sequence of junction points from T_L, where $\omega_{j+1} \geq \omega_j$, $j = 0, 1, 2, \dots$. Note that this array is sorted on the location of the junction points in ascending order except that the lower bound T_L may not be a junction point. Importantly, Corollary 2 asserts that the vector of optimal multipliers for the $\mathrm{TC}_{\mathrm{opt}}(T)$ function is invariant between ω_{j+1} and ω_j. Therefore, $F(\omega_j)$ is the vector of optimal multipliers for the $\mathrm{TC}_{\mathrm{opt}}(T)$ function in the interval $(\omega_{j+1}, \omega_j]$. Denoted as $\breve{T}(F(\omega_j))$ the minimum for the vector of multipliers $F(\omega_j)$, the following proposition indicates the existing condition and the location of a local minimum for the $\mathrm{TC}_{\mathrm{opt}}(T)$ function.

PROPOSITION 4. *Let*

$$\breve{T}(F(\omega_j)) = \sqrt{\frac{\left(K_p + K_m n_m p \Big/ \sum_{i=1}^{N} d_i + \sum_{i=1}^{N} (K_b/f_i^*(\omega_j))\right)}{\sum_{i=1}^{N} d_n \left[C_1 + C_2 f_i^*(\omega_j)\right]}}. \tag{40}$$

$\breve{T}(F(\omega_j))$ *is a local minimum for the* $\mathrm{TC_{opt}}(T)$ *function if* $\breve{T}(F(\omega_j)) \in (\omega_{j+1}, \omega_j]$ *where* $f_i^*(\omega_j) \in F(\omega_j)$, $\forall i$.

PROOF. For any given vector of $(f_1^*, \dots, f_N^*)$, one may obtain its local minimum, $\breve{T}(f_1^*, \dots, f_N^*)$, by taking the derivative of the $\mathrm{TC_{opt}}(T)$ function with respect to T, and equating it to zero. By Corollary 2, $\breve{T}(f_1^*, \dots, f_N^*)$ becomes a local minimum for the $\mathrm{TC_{opt}}(T)$ function when $\breve{T}(F(w_j)) \in (\omega_{j+1}, \omega_j]$. ∎

3.3. The Upper Bound

In order to obtain the optimal solution, the search scheme needs to secure all the local minima in $[T_{\mathrm{cc}}^*, T_U]$ where T_U is the upper bound to be derived in this section. Recalling that T_{cc}^* is the optimal replenishment cycle for the CC approach, we can obtain the corresponding vector of optimal multipliers $F(T_{\mathrm{cc}}^*)$ at T_{cc}^* by Corollary 1.

Let T^* and F^* be the best-on-hand local minimum and its corresponding vector of optimal multipliers, respectively, and we derive an upper bound β in Lemma 3. We note that our upper bound β is derived by asserting that for $T > \beta$, there exists no solution with its objective function value lower than $\mathrm{TC}(F(T^*), T^*)$.

LEMMA 3. *At the local minimum* T^*, *one may secure an upper bound* β *on the search range by*

$$\beta = \frac{\left(X + \sqrt{X^2 - 4\left(K_p + K_m n_m p \Big/ \sum_{i=1}^{N} d_i\right)\left(\sum_{i=1}^{N} d_i C_1\right)}\right)}{2 \sum_{i=1}^{N} d_i C_1}, \tag{41}$$

where

$$X = \frac{K_p + K_m n_m p \Big/ \sum_{i=1}^{N} d_i}{T^*} + \sum_{i=1}^{N} \phi_i\left(F\left(T^*\right), T^*\right) + T^* \sum_{i=1}^{N} d_i C_1, \tag{42}$$

and

$$\phi_i\left(f_i^*\left(T^*\right), T^*\right) = \begin{cases} \dfrac{K_b}{T^*} + d_i C_2 T^* - 2\sqrt{K_b d_i C_2}, & f_i^*\left(T^*\right) = 1, \\ \sqrt{K_b d_i C_2}\left(\dfrac{\left(2 + f_i^*\left(T^*\right)\right)}{\sqrt{1 + f_i^*\left(T^*\right)}} - 2\right), & f_i^*\left(T^*\right) < 1. \end{cases} \tag{43}$$

PROOF. We note that the function $\phi_i(f_i^*(T^*), T^*)$ indicates the maximum magnitude of decrement in $\mathrm{TC}_i(f_i, T)$ from T^* to any value of $T > T^*$ for buyer i. Recall that Lemma 1 asserts that the function $\mathrm{TC}_i(f_i, T)$ is bounded from below by $2\sqrt{K_b d_i C_2}$. If the optimal multiplier for buyer i is $f_i^*(T^*) = 1$, then the maximum magnitude of decrement in $\mathrm{TC}_i(f_i, T)$ from T^* to any value of $T > T^*$ is bounded by $K_b/T^* + d_i C_2 T^* - 2\sqrt{K_b d_i C_2}$. If $f_i^*(T^*) < 1$,

$$\mathrm{TC}_i\left(f_i^*\left(T^*\right), T^*\right) \le \max\left\{\mathrm{TC}_i\left(f_i^*\left(T^*\right), \delta_n\left(\frac{1 + f_i^*\left(T^*\right)}{f_i^*\left(T^*\right)}\right)\right), \mathrm{TC}_i\left(f_i^*\left(T^*\right), \delta_n\left(\frac{1}{f_i^*\left(T^*\right)}\right)\right)\right\}$$

by the piece-wise convexity of the $\mathrm{TC}_i(f_i^*(T^*), T^*)$ function, then, $\mathrm{TC}_i(f_i^*(T^*), T^*) \le \mathrm{TC}_i(f_i^* \cdot (T^*), \delta_i(1/f_i^*(T^*)))$, since one can easily prove that $\mathrm{TC}_i(f_i^*(T^*), \delta_i(1/f_i^*(T^*))) > \mathrm{TC}_i(f_i^*(T^*), \delta_i((1 + f_i^*(T^*))/f_i^*(T^*)))$. By plugging equation (34), we have the following concise expression for $\mathrm{TC}_i(f_i^*(T^*), \delta_i(1/f_i^*(T^*)))$ after some simplification.

$$\mathrm{TC}_i\left(f_i^*(T^*), \delta_i\left(\frac{1}{f_i^*(T^*)}\right)\right) = \sqrt{K_b d_i C_2}\left[\frac{(2 + f_i^*(T^*))}{\sqrt{1 + f_i^*(T^*)}}\right]. \tag{44}$$

In other words, if $f_i^*(T^*) < 1$, the maximum magnitude of decrement in $\mathrm{TC}_i(f_i, T)$ from T^* to any value of $T > T^*$ is bounded by $\sqrt{K_b d_i C_2}[(2+f_i^*(T^*))/\sqrt{1+f_i^*(T^*)}-2]$. Also, a set-up cost k_0 at warehouse would decrease from $(K_p + K_m n_m p/\sum_{i=1}^N d_i)/T^*$ to $(K_p + K_m n_m p/\sum_{i=1}^N d_i)/T$ from T^* to any value of $T > T^*$.

On the other hand, the sum of the holding cost for all the buyers would increase from $T^* \sum_{i=1}^N d_i C_1$ to $T \sum_{i=1}^N d_i C_1$ from T^* to any value of $T > T^*$.

The upper bound is derived by asserting that for $T > \beta$, the increment in the sum of the holding cost for all the buyers, i.e., $T\sum_{i=1}^N d_i C_1 - T^* \sum_{i=1}^N d_i C_1$, must exceed the maximum magnitude of decrement, i.e.,

$$\frac{\left(K_p + K_m n_m p \Big/ \sum_{i=1}^N d_i\right)}{T^*} - \frac{\left(K_p + K_m n_m p \Big/ \sum_{i=1}^N d_i\right)}{T} + \sum_{i=1}^N \phi_i\left(f_i^*(T^*), T^*\right);$$

or

$$\sum_{i=1}^N d_i C_2 (T - T^*) \geq \left(\frac{K_p + K_m n_m p}{\sum_{i=1}^N d_i}\right)\left(\frac{1}{T^*} - \frac{1}{T}\right) + \sum_{i=1}^N \phi_i\left(f_i^*(T^*), T^*\right),$$

which gives exactly equation (41). ∎

By using Lemma 3, we shall try to revise the upper bound of the search range as we obtain an updated best-on-hand solution during the search. We note that revising the upper bound may significantly improve the efficiency of the search algorithm since it could notably shorten the search range. (Please refer to the demonstrative example discussed in Section 4.1 for instance.)

3.4. The Algorithm

We are now ready to enunciate the proposed search algorithm. Recall that the algorithm searches from T_{cc}^* toward larger values of T until it meets the upper bound T_U. In the search process, we use a sequence of (sorted) junction points as the backbone and obtain all the local minima of the $\mathrm{TC}_{\mathrm{opt}}(T)$ function between $[T_{\mathrm{cc}}^*, T_U]$. Recalling that T^* and F^* the best local minimum and its corresponding vector of optimal multipliers, respectively, we summarize the step-by-step procedure of the proposed search algorithm as follows.

STEP 1. Set n_m=1 and $\mathrm{TC}^{\mathrm{BOH}} = \infty$.

STEP 2. The initialization.

- (a) Obtain the lower bound $T_L = T_{\mathrm{cc}}^*$ by equation (37). Then, use T_{cc}^* to obtain T_U by equation (37). Set $T_c = T_L$.
- (b) Calculate and sort all the junction points in equation (34). Set $w_1 = \delta_\pi(1/f_\pi)$ by $\pi = \arg\min_i\{\delta_i(1/f_i^*) > T_c\}$. Use Corollary 1 to obtain $F(T_L)$.
- (c) Check by Proposition 4: if $\breve{T}(F(T_L)) \in [w_2, w_1)$, set $T^* = \breve{T}(F(T_L))$ and $F^* = F(T_L)$, calculate $\mathrm{TC}(F^*, T^*)$. Also, set $j = 1$ and $T_c = w_j$.

STEP 3. The search procedure.

- (a) Obtain $F(w_j)$ by $F(w_j) \equiv (F(T_c) \setminus \{f_\pi^*\}) \cup \{f_\pi^*/f_\pi^* + 1\}$ and $w_{j+1} = \delta_\pi(1/f_\pi^*)$ by $\pi = \arg\min_i\{\delta_i(1/f_i^*) > T_c\}$.
- (b) Check by Proposition 4; if $\breve{T}(F(w_j)) \in [w_j, w_{j+1})$, calculate $\mathrm{TC}(F(w_j), \breve{T}(F(w_j)))$.
- (c) If $\mathrm{TC}(F(w_j), \breve{T}(F(w_j))) < \mathrm{TC}(F^*, T^*)$, set $T^* = \breve{T}(F(w_j))$ and $F^* = F(w_j)$.

STEP 4. The termination condition of the search algorithm.

(a) If $w_{j+1} > T_U$, output T^*, F^*, $\mathrm{TC}(F^*, T^*)$ and the algorithm stops the search for the current value of n_m.
(b) Otherwise, set $j = j + 1$ and $T_c = w_j$. Go to Step 3.

STEP 5. Try to revise the best-on-hand optimal solution. If $\mathrm{TC}^{\mathrm{BOH}} > \mathrm{TC}(F^*, T^*, n_m)$, then we update the best-on-hand solution by setting $T^{\mathrm{BOH}} = T^*$, $F^{\mathrm{BOH}} = F^*$, and $\mathrm{TC}^{\mathrm{BOH}} = \mathrm{TC}(F^*, T^*, n_m)$, and go to Step 6; otherwise, we terminate the algorithm.

STEP 6. $n_m = n_m + 1$; Go to Step 2.

4. NUMERICAL EXPERIMENTS

In this section, we present an example to demonstrate the implementation of the proposed search algorithm. Also, by using random experiments, we will show that the proposed search algorithm outperforms Yang and Wee's [1] heuristic.

4.1. A Demonstrative Example

Here, we use an example with three buyers to demonstrate the implementation of the proposed search algorithm. Then, by using the same example, we will show how the setting of the range of n_i-value in Yang and Wee's [1] heuristic could be a crucial factor in obtaining an optimal solution though they did not clearly specify how to set the range of n_i-value.

We employ the test data presented in Yang and Wee's [1] paper, but increase the number of buyers to three. We present the set of parameters used in this numerical example in Table 1.

Given different values of n_m, we solve the optimal solution for problem (P), and we plot the optimal objective function value versus n_m for this example in Figure 5. We note that the envelop of the optimal objective function value of problem (P) with respect to n_m is convex. Recall that the termination condition of our search algorithm is devised based on the convexity characteristic as shown in Figure 5.

Table 1. The set of parameters used in the demonstrative example.

		1	2	3
Buyer	Demand rate	12000	6000	2000
	Ordering cost	10		
	Holding cost	0.35		
	Price	24		
Producer	Setup cost	150		
	Holding cost	0.15		
	Price	20		
	Production rate	24000		
	Deterioration rate	0.1		
Raw material	Setup cost	5		
	Holding cost	0.15		
	Price	10		
	Deterioration rate	0.05		
	Unit usage of material	2.1		

Since we obtain the optimal solution for this example as $n_m = 6$, we summarize our search process only for the case of $n_m = 6$ as follows.

1. We first compute $T^*_{cc} = 0.03606$ by equation (37), and let $T_c = T^*_{cc}$. We obtain the vector of optimal multipliers at T^*_{cc}, i.e., $F(T^*_{cc})$, by $(1/3, 1, 1/2)$, and $\mathrm{TC}(F(T^*_{cc}), T^*_{cc}) =$ \$10, 271.28. We regard it as the best-on-hand objective value. We obtain the upper bound $T_U = 0.19209$ by equation (37).
2. We obtain $\omega_1 = \delta_1(1/f^*_1 = 1/3) = 0.0395$ by $\pi = \arg\min_i\{\delta_i(1/f^*_i) > T_c\} = 3$. By Proposition 4, we have local minimum $\breve{T}(F(T^*_{cc}))$ by 0.0507; since $\breve{T}(F(T^*_{cc})) \notin [T^*_{cc}, \omega_1)$, we do not have to revise the best-on-hand solution.
3. Next, we move to ω_1 and obtain the vector of optimal multipliers $F(\omega_1)$ by $F(\omega_1) \equiv F(T^*_{cc}) \setminus \{f^*_1 = 1/3\} \cup \{f^*_1 = 1/4\}$, which is given by $(1/4, 1, 1/2)$. Then, we let $T_c = \omega_1$ and secure $\omega_2 = \delta_2(1/f^*_2 = 1) = 0.03951$ by $\pi = \arg\min_i\{\delta_i(1/f^*_i) > T_c\} = 4$. We obtain the local minimum $\breve{T}(F(\omega_1))$ by 0.0536. Since the local minimum is not located in the interval $\breve{T}(F(\omega_1)) \notin [\omega_1, \omega_2) = [0.03950, 0.03951]$, we do not revise the best-on-hand solution.
4. As we move to ω_6 and obtain the vector of optimal multipliers $F(\omega_6)$ by $F(\omega_6) \equiv F(T^*_5) \setminus \{f_1 = 1/5\} \cup \{f_1 = 1/6\}$, which is given by $(1/6, 1/2, 1/4)$. Then, we let $T_c = \omega_1 = 0.0625$ and secure $\omega_7 = \delta_2(1/f_2 = 1/3) = 0.0685$ by $\pi = \arg\min_n\{\delta_n(1/f_n) > T_c\} = 2$. We obtain the local minimum $\breve{T}(F(\omega_6))$ by 0.0676.
5. The local minimum is located in the interval $\breve{T}(F(\omega_6)) \in [\omega_6, \omega_7) = [0.0625, 0.0685]$. We have $F^* = F(\omega_6) = (1/6, 1/2, 1/4)$, $T^* = \breve{T}(F(\omega_6)) = 0.0676$, and $\mathrm{TC}(F^*, T^*) =$ \$9046.87. Since it is less than \$10271.28, we thus revise the best-on-hand solution. We also revise the upper bound $T_U = 0.16549$ by equation (37).
6. As we move to ω_7 and obtain the vector of optimal multipliers $F(\omega_7)$ by $F(\omega_7) \equiv F(T^*_6) \setminus \{f_2 = 1/2\} \cup \{f_2 = 1/3\}$, which is given by $(1/6, 1/3, 1/4)$. Then, we let $T_c = \omega_7 = 0.0685$ and secure $\omega_8 = \delta_3(1/f_3 = 1/5) = 0.0722$ by $\pi = \arg\min_n\{\delta_n(1/f_n) > T_c\} = 3$. We obtain the local minimum $\breve{T}(F(\omega_7))$ by 0.0699.
7. The local minimum is located in the interval $\breve{T}(F(\omega_7)) \in [\omega_7, \omega_8) = [0.0685, 0.0722]$. We have $F^* = F(\omega_7) = (1/6, 1/3, 1/4)$, $T^* = \breve{T}(F(\omega_7)) = 0.0699$, and $\mathrm{TC}(F^*, T^*) =$ \$9045.66. Since it is less than \$9046.87, we thus revise the best-on-hand solution. We again revise the upper bound $T_U = 0.16518$ by equation (37).
8. As we continue the search, the next local minimum is secured in the interval $[\omega_9, \omega_{10}) = [0.0740, 0.0854]$. We have $F^* = F(\omega_9) = (1/7, 1/3, 1/5)$, $\breve{T}(F(\omega_9)) = 0.0742$, and $\mathrm{TC}(F, \bar{T})$ = \$9053.43. Since it is larger than \$9045.66, we therefore retain the last best-on-hand solution. Note that we do not revise the upper bound this time since the new $\beta = 0.16535$ by equation (37) is larger than the one on hand (0.16518).
9. We continue the search, but find no more local minimum less than \$9045.66. The search algorithm stops when it encounters the local minimum that is larger than T_U. In this example, before the search algorithm terminates, it visits totally 90 junction points. Note that if we did not revise the upper bound and simply use the upper bound obtained at T^*_{cc}, i.e., $T_U = 0.19209$, we need to visit totally *126* junction points. Our upper bound revising technique assists the search algorithm to save around 25% of the run time in this example.
10. For the entire search process, we secure only three local minima for this example. All of the local minima, their corresponding vector of optimal multipliers, and their objective function values are summarized in Table 2.

Table 2. The local minima obtained in the search process of the proposed search algorithm.

$[\omega_j, \omega_{j+1})$	f_1	f_2	f_3	$\breve{T}_j$	$\mathrm{TC_{opt}}\left(\breve{T}_j\right)$	On-Hand T_U	β	Revised T_U
[0.0625, 0.0685)	6	2	4	0.0676	$9046.87	0.19209	0.16549	0.16549
[0.0685, 0.0722)	6	3	4	0.0699	$9045.66	0.16549	0.16518	0.16518
[0.0740, 0.0854)	7	3	5	0.0742	$9053.43	0.16518	0.16535	0.16518

By the proposed algorithm, we obtain the optimal solution and the vector of optimal multipliers as $T^* = 0.0699$ and $F^* = (1/6, 1/3, 1/4)$, respectively, with the optimal objective value being $9045.66.

For the rest of this section, we will show how the setting of the maximum n_i-value (equivalently, the range of n_i-value) could significantly affect the solution quality of Yang and Wee's [1] heuristic. We note that one may encounter two possible problems from setting the maximum n_i-value; namely,

(1) one could miss the optimal solution by underestimating the maximum n_i-value, or,
(2) one may have an overflow problem when running Yang and Wee's heuristic by overestimating the maximum n_i-value.

In the following experiments, we test Yang and Wee's heuristic by different settings of the maximum n_i-value as follows.

1. If we set the maximum n_i-value to be 3, there are totally 9 ($= 3^3$) potential solutions that need to be tested. However, we obtain no feasible solution by Yang and Wee's heuristic.
2. As we set the maximum n_i-value to be 4, there are totally 64 ($= 4^3$) potential solutions that need to be tested. Again, we obtain no feasible solution by Yang and Wee's heuristic.
3. When we set the maximum n_i-value to be 5, there are totally 125 ($= 5^3$) candidate solutions. We secure the solution and the set of multipliers as $T^* = 0.0607$ and $F^* = (1/3, 1/3, 1/4)$, respectively. The optimal $n_m = 5$ and its optimal objective value $9417.48 is 4.1% larger than ours (i.e., $9045.66).
4. If we set the maximum n_i-value to be 8, which is larger than the maximum multiplier that we obtained in the proposed search algorithm, then totally 2,187 ($= 3^7$) potential solutions should be tested, and Yang and Wee's heuristic obtains a solution as above.

As one may observe, the number of the candidate solutions grows extremely fast as the setting of the maximum n_i-value and the number of buyers increase. Imaging that, if there are seven buyers involved in this system and the maximum n_i-value is 8, we would have totally 2,097,152 ($= 8^7$) potential solutions to be tested. In order to prevent missing the optimal solution, one needs to set a larger value for the maximum n_i-value. However, it requests a price of spending a significant amount of computational efforts, but possibly, without guaranteeing to obtain the optimal solution by using Yang and Wee's heuristic. Also, when using large values for the maximum n_i-value, we have often encountered an overflow problem since the enumeration load of Yang and Wee's heuristic exceeds the capacity of the personal computer during our experiments.

4.2. Random Experiments

In this section, we present our random experiments to show the proposed search algorithm outperforms Yang and Wee's [1] heuristic. The annual demand rates and the production rate were

randomly generated from uniform distributions UNIF[2,000–12,000] and UNIF[240,000–400,000], respectively. The holding costs for the vendor, the buyers, and raw materials were randomly generated from uniform distributions UNIF[0.15–0.20], UNIF[0.35–0.40], and UNIF[0.15–0.20], respectively. On the other hand, we borrow the other parameters, such as the ordering cost for

Table 3. The comparison between the proposed search algorithm and Yang and Wee's [1] heuristic.

Number of Buyers	Production Setup Cost	Max Error (%) in Y&W	Min Error (%) in Y&W	Avg. Error (%) Y&W	Run Time of H&Y	Run Time of Y&W	Max. of n_i
2	150	4.01	2.40	3.42	3.00	0.25	8
	250	4.82	2.95	4.11	3.50	0.25	9
	350	5.58	3.63	4.86	5.25	0.25	11
	450	5.90	4.25	5.09	4.77	0.25	12
	550	6.22	4.07	5.55	6.00	0.25	13
4	150	3.79	2.97	3.32	1.75	0.25	5
	250	6.48	4.72	5.79	2.75	0.25	7
	350	7.76	5.34	6.42	4.52	0.25	9
	450	8.02	5.45	6.82	5.75	3.50	9
	550	8.39	5.81	6.93	7.25	7.00	11
6	150	3.56	1.89	2.50	2.25	3.75	5
	250	5.65	4.37	4.83	3.77	3.77	6
	350	7.47	5.43	6.46	5.75	3.75	7
	450	8.32	5.18	7.02	7.25	3.75	8
	550	8.72	4.97	6.74	9.27	3.75	10
8	150	2.57	1.13	1.81	2.75	11.77	5
	250	4.40	3.37	3.88	4.75	11.52	5
	350	6.67	5.24	5.91	6.77	11.50	6
	450	7.89	5.29	6.87	9.25	11.02	7
	550	8.81	4.46	7.20	11.77	12.27	8
10	150	2.20	0.41	1.22	3.00	161.97	4
	250	3.63	2.58	3.02	5.50	161.22	5
	350	5.96	4.25	4.91	8.52	161.50	5
	450	6.71	4.64	5.87	11.52	161.47	6
	550	8.26	5.13	6.81	17.27	161.22	7

the buyers, the deterioration rate and the unit price for the vendor and the buyers, from the example in Yang and Wee's [1] paper. Also, following the same assumptions stated in Yang and Wee's [1] paper, we set C_b and C_bF_b to be larger than C_p and C_pF_p, respectively. We tested five settings of the number of buyers ($N = 2, 4, 6, 8, 10$) and five settings of the setup cost for the vendor ($K_p = 1500, 2500, 3500, 4500, 5500$). For each combination of N and K_p, we randomly generated 500 examples and solved each of them by the proposed search algorithm and Yang and Wee's heuristic. We summarize the comparison of these two solution approaches in Table 3 in which we use H&Y and Y&W to represent the proposed search algorithm and Yang and Wee's heuristic, respectively.

We have some observations from Table 3 as follows. First, when the number of buyers is not large, the run time of Yang and Wee's heuristic is very short, but its solution quality is not as good as the proposed search algorithm. When the number of the buyers is larger, the proposed search algorithm becomes more efficient than Yang and Wee's heuristic (in its run time). Second, the error percentage of Yang and Wee's solutions increases as the value of the vendor's setup cost K_p increases.

Based on our random experiments in Table 3, we conclude that the proposed search algorithm outperforms Yang and Wee's [1] heuristic.

5. CONCLUDING REMARKS

In this study, we focus on optimally determining lot-sizing policies for a deteriorating item among all the partners in a supply chain system so as to minimize the average total costs. We revise Yang and Wee's [1] model using the Fourier series to precisely estimate the vendor's inventory holding costs. Also, we transform our revised model into a more concise version by applying an approximation to the exponential terms in the objective function. In order to solve this problem, we analyze the optimality structure of our revised model and derive several interesting properties. By utilizing our theoretical results, we propose a search algorithm that can efficiently solve the optimal solution. Based on our numerical experiments, we show that the proposed algorithm outperforms the existing solution approach in the literature, especially when the number of buyers is larger in this supply chain system.

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