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The solutions of time-delayed optimal control problems by the use of modified line-up competition algorithm

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ABSTRACT

In this article, line-up competition algorithm (LCA), a brand-new nonlinear programming method based on the principle of evolution, is applied to solve time-delay optimal control problems (TDOCPs). The problems are first discretized based on the concept of control vector parametrization, and then solved by LCA. Since the delay differential equations are directly integrated without using the auxiliary procedures such as model conversion and data interpolation, most TDOCPs can be solved very conveniently under such solution framework. Meanwhile, a more efficient sampling strategy is adopted to promote the too slow convergence of the basic LCA. By solving six typical examples, including five pure mathematical problems and one chemical engineering problem, the modified LCA demonstrates a robust and efficient property in optimizing time-delay unsteady systems.

1. Introduction

For fast adapting to the changes of market demand, high-priced chemical commodities are often produced by unsteady (or batch) processes. Without a fixed set-point to regulate the deviations through proportional–integral—derivative (PID) controller loops, these unsteady systems often rely on the manipulations of experienced workers. In recent years, digital computers are widely applied to design, operation and control of chemical processes. Using numerical optimization techniques to off-line determine unsteady operating policy, the so-called *optimal control problem* (OCP), and then integrating the obtained policy into distributed control systems (DCS) have been a very promising way toward the automation of unsteady chemical systems. Owing to the highly nonlinear, multi-modal and discontinuous natures embedded in most chemical systems, to determine such an optimal input policy is in general very challenging. Moreover, derived from distributed nature or delayed elements in engineering systems, for example recycle stream, transportation and measurement lag, time delay (or time lag) is also a common phenomenon that interferes the operations of unsteady chemical systems. The existence of time delay implies the process outputs are unable to respond to the inputs immediately. Thus, ignoring such factor embedded in unsteady chemical systems must significantly affect the feasibility of the obtained operating policy. Therefore, how to efficiently solve time-delay optimal control problems (TDOCPs) is very meaningful for chemical community.

Concerning the solution of time-delay optimal control problems (TDOCPs), Banks and Burns (1978, 1979) first use the averaging approximation to deal with the past history of linear time-delay systems. The resultant problems which have the same size as the original ones can be further solved by the use of the minimum principle. Later, such solution procedure is extended to the solution of nonlinear time-delay systems. Wong *et al.* (1985) propose solving a series of delay-free problems with a set of piecewise constant input by using the second kind Oguztöreli kernel matrix to convert time-delay states. Based on the Páde approximation, Lee (1993) uses the auxiliary state variables to estimate time-delay terms. As a result, the augmented models can be optimized by any well-established technique. Actually, such a procedure is very indirect. Even the approximation errors can be reduced by adding more state variables, the computational times spend in the numerical integrations must be increased considerably. Especially, all above methods only acquire approximate solutions. With the advances of the numerical integrators for delay differential equations (DDEs), Dadebo and Luus (1992) and Dadebo and McAuley (1995) suggest directly applying the model equations to the solution of TDOCPs by iterative dynamic programming (IDP). Owing to the backward solution manner, however, IDP still requires to interpolate the data from the last iteration to estimate the historical information of time-delay states. To avoid the data interpolation, Chen *et al.* (2000) use the Taylor's expansion to

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Nomenclature

$\mathbf{f}$	right-hand side ordinary differential equations
$\mathbf{L}^0$	the lower bound of the decision vector
$\mathbf{u}$	control vector
$\mathbf{U}^0$	the upper bound of the decision vector
g	inequality constraint
h	equality constraint
$\mathcal{I}[x(t_f)]$	performance index
L_i^0	the lower bound for the i-th decision variable
N_f	numbers of family in one generation
N_g	numbers of generation
N_m	numbers of member in a family
N_P	numbers of dimension of control vector
P	numbers of time division from t_0 to t_f
t_0, t_f	initial and final time
U_i^0	the upper bound for the i-th decision vector

Greek symbols

β	region contraction factor
γ_i	pseudo-random number from a normal distribution of zero mean and one standard deviation
$\Delta\omega_i$	search region for the i-th time grid
ε_i	tolerance factor
ζ	uniform random number ranging from 0 to 1
λ	the heuristic parameter with the value of 1/3 or lower
ξ_i	i-th discretized control variable
Ξ	measure function for final state constraint
σ_c	the distance from the current best variable to its nearest bound
φ_i	the i-th decision variable in Eq. (5d)
Φ	decision vector in Eq. (5d)
Ψ	measure function for path constraint
ω_i	i-th time grid

estimate time-delay state variables under the framework of IDP. However, both procedures derived from IDP often rely on a large number of time and state grids to refine the solutions. Consequently, the enlarged problem size often leads to a slow convergence. In recent years, using nonlinear programming (NLP) techniques to solve delay-free OCPs under the framework of control vector parametrization becomes very popular. The solution procedures for such kind of methods are in general divided into outer and inner loops. In outer loop, objective value and/or gradient are evaluated with the help of numerical integrators. These results are then passed to the inner loop to further optimize the parameterized controls (decision variables). In general, the gradient-based optimizers may converge the solutions of delay-free OCPs very rapidly. Unfortunately, such kind of optimizers are unable to apply in the solutions of TDOCPs owing to the associated discontinuities. As a result, the direct methods that optimize the parameterized controls based on whether the objective value is improved or not become the only way to solve time-delay optimal control problems. Even such kind of methods are conceptually very straightforward, rare papers explore such applications in our reviews.

As a special class of direct optimizers, evolutionary algorithms (EAs) including genetic algorithm, evolutionary strategy and evolutionary programming are very popularly used in recent to solve difficult optimization problems. All these methods are based on the principle of evolution in nature. The major differences between them are that genetic algorithm stresses gene operations, whereas evolutionary strategy and evolutionary programming focus on the behavior change of individuals (Yan and Ma, 2001). In past decade, many papers (Babu and Angira, 2006; Michalewicz *et al.*, 1992; Simant and Upreti, 2004; Wang and Chiou, 1997) discuss the use of EAs to the solutions of delay-free optimal control problems. However, because the diversity of population and the balance of local and global searches are hard to maintain, the aforementioned EAs often demonstrate slow convergence, and even get local solutions in some cases. To improve these drawbacks, Yan and Ma (2001, 2003) and Yan *et al.* (2004) propose the line-up competition algorithm (LCA) based on the concept of cooperation and competition among biological populations. In basic LCA, independent and parallel evolutions are carried out in sub-regions. Since these regions are different in size, the trial vectors that are randomly generated in different sub-regions have diverse driving force to propel mutation. Thus, LCA may produce the population with higher diversity than common EAs. Through competition and cooperation among these population, the global optimum can be approached very rapidly. By testing various static optimization problems, the method has been proved to be very robust and efficient. However, when the basic LCA is applied to the solution of delay-free OCPs, the convergent quality becomes very unsatisfactory (Sun *et al.*, 2007). Therefore, the authors suggest introducing normal (or Gaussian) sampling policy to replace uniform sampling policy originally used in the basic LCA. Compared with other EAs, such modification is confirmed to be very efficient in converging the solutions to the vicinity of global optimum within 1%.

In this study, we use well-developed delay differential equations (DDEs) integrator as inner solver coupled with the modified LCA as outer optimizer to solve TDOCPs under the framework of control vector parametrization. Because the objective value can be directly determined by the integration of DDEs, the procedures for approximating time-delay states are basically unneeded unless the problems are very highly nonlinear. Compared with the existent methods, the proposed procedure may greatly simplify the solution of TDOCPs, and quickly approach to the vicinity of the global optimum. In the rest of this article, Section 2 presents the formulation of time-delay optimal control problems. Section 3 briefly introduces the line-up computation algorithm. The improvement of LCA and the detailed computational steps are also presented in this section. Section 4 discusses how to discretize a TDOCP based on control vector parametrization and how to rebuild continuous input by given parameters. The numerical examples are illustrated in Section 5. Finally, the conclusions are made in Section 6.

2. Problem formulation

Let us consider an unsteady system described by a set of ordinary differential equations with constant time delay τ,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau), \mathbf{u}, t) \tag{1a}$$

with the initial conditions

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{1b}$$

and the initial state profile

$$\mathbf{x}(t) = \psi(t) \quad -\tau \le t < 0 \tag{1c}$$

where $\mathbf{x}$ and $\mathbf{u}$ respectively denote $(n \times 1)$ state vector and $(m \times 1)$ control inputs physically restricted by

$$\underline{\mathbf{u}} \le \mathbf{u} \le \bar{\mathbf{u}} \tag{1d}$$

In the field of chemical engineering, Eq. (1) are in general derived from mass and/or energy balance relations. Therefore, the aim of this problem is to find the continuous $\mathbf{u}(t)$ over $t \in [0, t_f]$, by which one of the following objective function is minimized.

- *Bolza* type

$$\mathcal{I}_1[\mathbf{x}, \mathbf{u}] = \Theta[\mathbf{x}(t_f)] + \int_0^{t_f} \Omega(\mathbf{x}, \mathbf{u}, t)\mathrm{d}t \tag{2}$$

- *Lagrange* type:

$$\mathcal{I}_2[\mathbf{x}, \mathbf{u}] = \int_0^{t_f} \Lambda(\mathbf{x}, \mathbf{u}, t)\mathrm{d}t \tag{3}$$

- *Mayer* type:

$$\mathcal{I}_3[\mathbf{x}, \mathbf{u}] = \Lambda[\mathbf{x}(t_f)] \tag{4}$$

where t_f represents the fixed ending time. Notably, Eqs. (2) and (3) can be easily converted into Eq. (4). For instant, $\mathcal{I}_1$ can be rewritten as

$$\int_0^{t_f} \left\{ \Omega(\mathbf{x}, \mathbf{u}, t) + \frac{\mathrm{d}}{\mathrm{d}t}\Theta[\mathbf{x}(t)] \right\}\mathrm{d}t + \Theta[\mathbf{x}(t_0)]$$

Because the minimization of $\mathcal{I}_1$ does not affect $\Theta[\mathbf{x}(t_0)]$, we only consider the first term. By defining $\Lambda \equiv \Omega(\mathbf{x}, \mathbf{u}, t) + (\mathrm{d}/\mathrm{d}t)\Theta[\mathbf{x}(t)]$, the Bolza-type objective function is converted into Lagrange-type. Furthermore, the evaluation of Eq. (3) can be carried out by integrating the state equation

$$\dot{x}_{n+1} = \Lambda(\mathbf{x}, \mathbf{u}, t)$$

with the initial condition $x_{n+1}(0) = 0$. Thus, the conversion from Eqs. (3) and (4) is achieved. Without lose the generality, the Mayer-type objective function is adopted in further discussions. On the other hand, the objective function have to change its sign for maximization problems.

3. An overview of line-up competition algorithm

Used as the outer optimizer in solving TDOCPs, the line-up competition algorithm is originally devised to optimize the nonlinear, non-convex and multi-modal problems as follows

$$\mathcal{I}[\Phi] \tag{5a}$$

subject to

$$\mathbf{h}(\Phi) = 0 \tag{5b}$$

$$\mathbf{g}(\Phi) \le 0 \tag{5c}$$

$$\mathbf{L}^0 \le \Phi \le \mathbf{U}^0 \tag{5d}$$

where $\Phi \equiv [\varphi_1, \ldots, \varphi_{N_c}]'$ means the decision vector, $\mathbf{h}$ and $\mathbf{g}$ are the equality and inequality constraints, respectively. Furthermore, Φ is physically restricted by the lower and upper bounds, $\mathbf{L}^0 \equiv [L_1^0, \ldots, L_c^0]'$ and $\mathbf{U}^0 \equiv [U_1^0, \ldots, U_c^0]'$.

Firstly, the LCA uniformly generates N_f trial vectors over the entire search space. This first generation of trial vectors, named as fathers, are then queued in sequence, named as line-up, according to the corresponding objective values. The father having the best objective value is placed at the first position of this sequence, whereas the poorest is at the last place. Obviously, the line-up is raked in ascending order for a minimization problem, or in descending order for a maximization problem. So far, the first generation is completed. The size of search space is then reduced and systematically divided into N_f regions (or sub-spaces). Based on the corresponding position in the line-up, each father is allocated one of the sub-regions to further produce N_m trial vectors, named as "offsprings". The father and its offsprings consist of a *family*. Noted that the fathers placed in the front of line-up are given the smaller sub-regions, whereas the fathers at the rear positions are given the larger ones. Currently, the entire search space has been occupied by N_f families, each having $N_m + 1$ members. All the members in the same family also compete with each other based on the idea of "survival of the fittest". The family member having the best objective value is chose as the candidate to strive for a more prior position in the next line-up. The aforementioned procedures, including division, sequencing and selection, go further until a given generations.

From the above descriptions, the important features of LCA can be summarized as follows. First, each family takes charge of seeking the best solution in the given sub-space. Thus, through the competitions inside each family, the possible solutions spreading over the entire search space can be thoroughly examined. Second, the current better solutions can be effectively refined by allocating smaller sub-spaces to the front families. Meanwhile, the ability of the rear families to escape from the local sub-regions are also reinforced by giving larger search spaces. Third, with unceasingly adjusting the sizes of the sub-spaces during solutions, all the families are endowed with the missions to refine the current better solutions and to seek the new global one. Thus, the local convergence is able to be surpassed effectively, even some families gather around a local solution. As a result, the global solution can be quickly approached through the above competition and cooperation among families. Furthermore, the region contraction strategy also accelerate the rear families to converge the solution toward global optimum.

3.1. The improvement of LCA

Like most evolutionary algorithms, the original LCA evenly samples new trial members from each sub-region based on uniform distribution. Such sampling policy regards all trial vectors in the same search space as having identical probability to be selected. Therefore, it is theoretically possible to reach the global optimum, provided the numbers of sampled members are huge enough (Masri and Bekey, 1980). Although the uniform sampling policy is very exhaustive, it also faces the criticism of excessive objective function evaluations and too slow convergence. Such impact becomes more serious for high-dimensional problems. In the past, many strategies are taken to tackle the aforementioned problem. Luus and Jaakola (1973) introduce the concept of region reduction to enhance the refinement of current optimum, which is the so-called Luus–Jakkola optimization procedure. Evidently, the basic LCA has contained such concept. Applying the basic LCA is unable to fast converge the high-dimensional optimization problems, such as discretized delay-free OCPs. Therefore, based on the concept by Goulcher and Casares (1978), the authors suggest sampling the new trial members in each family from the neighborhood of the current best solution based on the standard normal (or Gaussian) distribution. As generally known, the normal distribution is a class of continuous probability distributions. The members in this class are defined by two parameters: mean and variance (or standard deviation squared). The standard normal distribution is a normal distribution with zero mean and unit variance. Once the trial members in LCA are selected based on the standard normal distribution, the father in each family is considered as the mean. The size of the variance for each family is calculated as the function of the father, low and upper bounds on the trial members and a heuristic parameter with the value of 1/3. Thus, since the new trial members around the father have more chances to be selected, the refinement of the solutions in prior families can be enhanced. For the solutions of rear families, the improvement can be intensified through competition and cooperation among the families. The detail computational steps for LCA are presented in the next.

3.1.1. The proposed algorithm

1. Assign the numbers of generation, member and family, namely N_g, N_m and N_f.
2. Set the generation counter g as 1. Uniformly generate N_f decision vectors

$$\left\{\Phi^{(*,f,g)} | f = 1, \ldots, N_f\right\} \tag{6}$$

as the fathers of the first generation over the entire searched space. The f th decision vector in Eq. (6) consists of N_c decision variables

$$\Phi^{(*,f,g)} = [\varphi_1^{(*,f,g)} \ldots \varphi_c^{(*,f,g)} \ldots \varphi_{N_c}^{(*,f,g)}]' \tag{7}$$

The value of $\varphi_c^{(*,f,g)}$ can be initially assigned, or produced by an uniform random number generator as follows

$$\varphi_c^{(*,f,g)} = L_c^0 + \zeta \cdot \Delta_c^{(f,g)} \tag{8}$$

where $\Delta_c^{(f,g)} = U_c^0 - L_c^0$ means the size of the search space for the c th decision variable; ζ is an uniform random number ranging from 0 to 1.
3. Calculate the corresponding objective value for each father.

$$\{(\Phi^{(*,f,g)}, \mathcal{I}^{(*,f,g)}) | f = 1, \ldots, N_f\} \tag{9}$$

4. (**Competitions between families**) According to the corresponding objective values, line up the fathers and the associated search spaces in ascending/descending order for minimization/maximization problems, namely

$$\begin{array}{c}\{(\Phi^{(*,f,g)}, \mathcal{I}^{(*,f,g)}, \Delta^{(f,g)}) | (f = 1, \ldots, N_f)\} \\ \downarrow \text{ordering} \\ \{(\tilde{\Phi}^{(*,f,g)}, \tilde{\mathcal{I}}^{(*,f,g)}, \tilde{\Delta}^{(f,g)}) | (f = 1, \ldots, N_f)\}\end{array}$$

where $\Delta^{(f,g)} \equiv [\Delta_1^{(f,g)}, \cdots, \Delta_{N_c}^{(f,g)}]'$ is the size vector of the f th decision vector, $\tilde{\Delta}^{(f,g)} \equiv [\tilde{\Delta}_1^{(f,g)}, \ldots, \tilde{\Delta}_{N_c}^{(f,g)}]'$ is the new size vector after ordering.
5. Reduce the search region of the worst family, $\tilde{\Delta}^{(N_f,g)}$ to act as the search region for the next generation

$$\Delta \leftarrow \beta \cdot \tilde{\Delta}^{(N_f,g)} \tag{10}$$

where the default value of β is 0.9.
6. Add 1 to the counter g. Calculate the new bounds and new searching space for each family in the next generation via Eq. (10).

$$\tilde{L}_c^{(f,g)} = \max\left\{\tilde{\varphi}_c^{(*,f,g)} - \left(\frac{f \cdot \Delta}{N_f}\right), L_c^0\right\} \tag{11}$$

$$\tilde{U}_c^{(f,g)} = \min\left\{\tilde{\varphi}_c^{(*,f,g)} + \left(\frac{f \cdot \Delta}{N_f}\right), U_c^0\right\} \tag{12}$$

$$\Delta_c^{(f,g)} = \tilde{U}_c^{(f,g)} - \tilde{L}_c^{(f,g)} \quad c = 1, \ldots, N_c \quad f = 1, \ldots, N_f \tag{13}$$

7. Generate N_m decision vector for each family. The c-th variable of the m-th vector in the f-th family is determined by

$$\varphi_c^{(m,f,g)} \leftarrow \tilde{\varphi}_c^{(*,f,g-1)} + \lambda\gamma\sigma_c \quad m = 1, \ldots, N_m \tag{14}$$

where σ_c is the distance from the current best variable $\tilde{\varphi}_c^{(*,f,g-1)}$ to its nearest bound, namely

$$\min\{(\tilde{U}_c^{(f,g)} - \tilde{\varphi}_c^{(*,f,g-1)}), (\tilde{\varphi}_c^{(*,f,g-1)} - \tilde{L}_c^{(f,g)})\} \tag{15}$$

γ is a pseudo-random number from a normal distribution of zero mean and one standard deviation; λ is the heuristic parameter with the value of 1/3.
8. (**Competitions between members**) Select the member with the best objective value from the f th family, namely

$$\{(\Phi^{(m,f,g)}, \mathcal{I}^{(m,f,g)}) | f = 1, \ldots, N_f\} \underset{Select}{\rightarrow} (\Phi^{(*,f,g)}, \mathcal{I}^{(*,f,g)}, \Delta^{(f,g)}) \tag{16}$$

To further elaborate the above descriptions, a simplified flow chart is shown in Fig. 1. The modified LCA has been proved to be very robust and efficient in solving delay-free OCPs (Sun *et al.*, 2007). The authors try to extend this algorithm to the solution of TDOCPs.

4. Control vector parametrization

To apply the line-up competition algorithm, the control input(s) in Eq. (1) are needed to be discretized, which is the so-called control vector parametrization. For simplifying the use of notations, the following formulation is restricted to the single-input systems. Thus, the dimension of control vector is reduced to 1, namely $u(t) \in R^1$. Such treatment is very straightforward to extend to multi-inputs systems. Consisting of Eqs. (4) and (1), a TDOCP is discretized into P consecutive sub-problems along with the time horizon from 0 to t_f

$$\min_{(\xi_1,\omega_1),\ldots,(\xi_{P+1},\omega_{P+1})} \mathcal{I} = \Lambda[\mathbf{x}(t_f)] \tag{17}$$

subject to

$$\begin{aligned}&\dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{x}(t-\tau), u_i(t), t) \\ &\mathbf{x}(t) = \psi(t) \quad -\tau \le t < 0 \\ &\mathbf{x}(0) = \mathbf{x}_0 \\ &\underline{u} \le u_i(t) \le \bar{u} \\ &\forall t \in [\omega_i, \omega_{i+1}) \quad i = 1, \ldots, P\end{aligned} \tag{18}$$

where ξ_i means the control at the i-th time grid ω_i; $u_i(t)$ denotes the control input applied in the duration of $[\omega_i, \omega_{i+1})$. Because ω_1 and ω_{P+1} are respectively corresponding to 0 and t_f, the goal of the above problem is stated as finding $2P$ parameters to minimize Eq. (17) subject to Eq. (18). Thus, a TDOCP with infinite dimension is converted into a finite-dimensional NLP problem, which can be further optimized by LCA.

In integrating the above DDEs, many mathematical expressions including step-type function (Vassiliadis *et al.*, 1994a,b), ramp-type function (Carrasco and Banga, 1997), wavelet and B-splines functions (Binder *et al.*, 2000; Schlegel *et al.*, 2005) are considered to evaluate the value of $u_i(t)$. Therein, step-type function and ramp-type function would be the most convenient forms from engineering points of view. However, TDOCPs contain intrinsic discontinuity as previously mentioned. Using the strategies of step-type function must bring new artificial discontinuities into solutions. Therefore, the policy of using ramp input with fixed time interval (RIFTI) is good for reducing the above difficulty. The expression for a RIFTI is described as

$$u_i(t) = \frac{\xi_{i+1} - \xi_i}{\omega_{i+1} - \omega_i}(t - \omega_i) + \xi_i \quad i = 1, \ldots, P+1 \tag{19}$$

where $\omega_i \equiv t_f(i-1)/P$. As a result, the time grids are excluded from the decision variables. The problem consisting of Eqs. (17) and (18) is recast as

$$\min_{\xi_1,\cdots,\xi_{P+1}} \mathcal{I} = \Lambda[\mathbf{x}(t_f)]$$

subject to Eq. (18) with RIFIT.

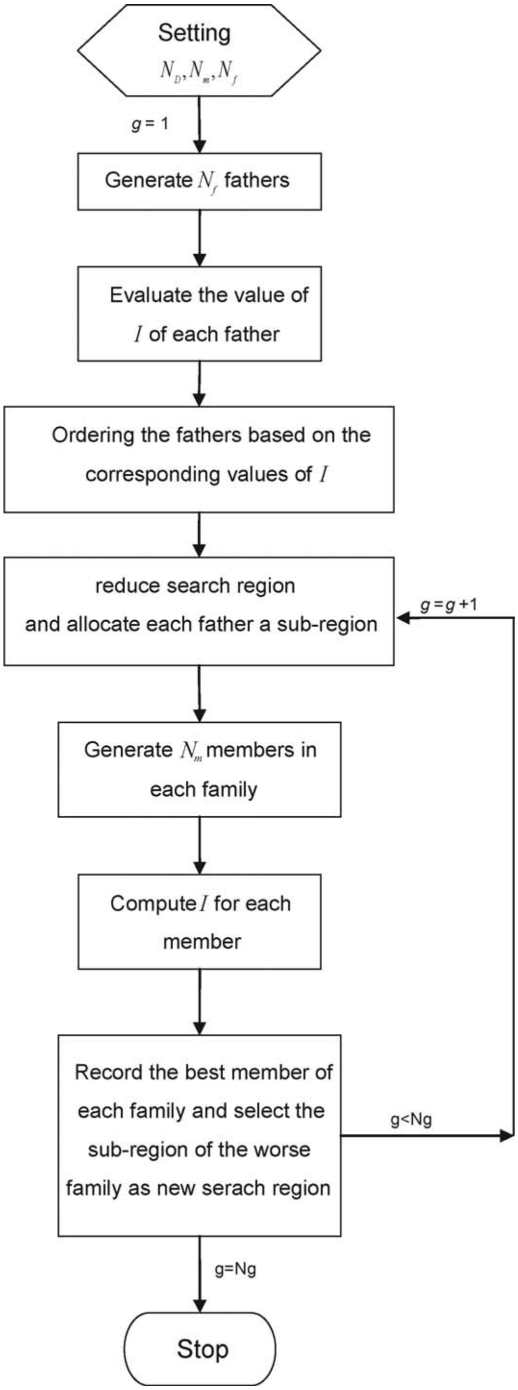

Fig. 1. The flow diagram of LCA.

Notably, many DDE solvers such as DDE (Paul, 1995), DKLAG6 (Corwin *et al.*, 1997) and DDVERK (Enright and Hayashi, 1997), have been proposed in recent. Based on continuous Runge-Kutta formula with defect control, DDVERK provides a very convenient interface to integrate with the proposed algorithm. We select DDVERK as the inner solver of this study to evaluate the objective functions. The scheme for solving a TDOCP by LCA under control vector parametrization is illustrated in Fig. 2.

5. Illustrated examples

In this section, six well-known problems are provided to demonstrate the proposed algorithm, which is coded by Compaq Visual Fortran 6.6a under Windows XP operating system. All the following computations are performed on the computer with Intel Core2 Quad Q6600 CPU and 2 Giga of random access memory (RAM).

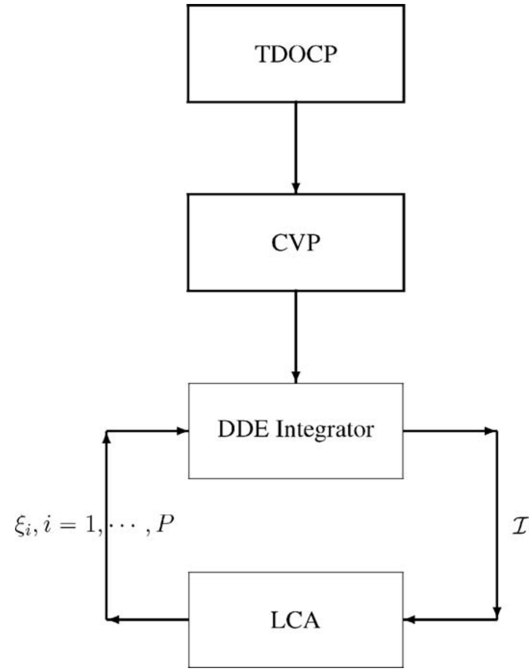

Fig. 2. A diagrammatic representation for solving time-delay optimal control problems by LCA.

Example 1. A single-input/two-outputs time-delayed system (Chan and Perkins, 1973; Dadebo and Luus, 1992; Oh and Luus, 1976).

The first problem consists of the following differential equations

$$\dot{x}_1 = x_2(t)$$
$$\dot{x}_2 = -10x_1(t) - 5x_2(t) - 2x_1(t-\tau) - x_2(t-\tau) + u(t)$$
$$\dot{x}_3 = \frac{1}{2}(10x_1^2(t) + x_2^2(t) + u^2(t))$$

with the initial conditions

$$\mathbf{x}_0 = [1. \quad 1. \quad 0.]'$$

and the initial profiles

$$x_1(t) = x_2(t) = 1.0 \qquad -1. \le t \le 0.$$

The objective function to be minimized is

$$\mathcal{I} = x_3(5)$$

Chan and Perkins (1973) convert this problem with $\tau = 1.0$ by a truncated Macluarin series approximation, then solve the converted problem by parameter embedding technique. Checking the solution by the truncated Taylor expansion and control vector iteration, Oh and Luus (1976) obtain the minimized objective value to be 2.932. By the use of two-pass IDP with 10 time divisions, Dadebo and Luus (1992) acquire the minimum objective value to be 2.953. This value is further reduced to 2.937 by 20 time divisions. Notably, the linear interpolation is incorporated into IDP to evaluate the value between the state grid and the delayed state corresponding to that particular grid.

Table 1
The objective values for Example (1) by LCA with $N_g = 50$ and $\beta = 0.9$.

P	N_f	N_m	Basic	Modified
5	3	16	2.9509	2.9484
	4	12	2.9868	2.9483
	6	8	2.9603	2.9487
	8	6	2.9696	2.9479
10	3	16	2.9952	2.9323
	4	12	2.9768	2.9321
	6	8	2.9499	2.9319
	8	6	2.9637	2.9321

Table 2
The objective values for Example 2 by LCA with $N_g = 50$ and $\beta = 0.8$.

P	N_f	N_m	Basic	Modified
5	4	20	2.9723	2.9482
	5	16	2.9838	2.9485
	8	10	2.9966	2.9485
	10	8	2.9761	2.9471
10	4	20	2.9854	2.9346
	5	16	2.9688	2.9338
	8	10	2.9917	2.9323
	10	8	2.9820	2.9326

In our solutions, the total trial members, $N_f \times N_m$, in one generation is firstly given as 48. The β factor is set as 0.9. As shown in Table 1, the objective values by the modified LCA after 50 generations are converged to the vicinity of 2.948 for $P = 5$, which are smaller than the solutions by the basic LCA. We further add the time divisions to 10. The modified LCA refines the objective values to the vicinity of 2.932, while the basic one brings no further improvements to the solutions. When the total trial members in one generation are increased to 80, and the β factor is reduced to 0.8 to accelerate the rate of region contraction. As shown in Table 2, the objective values by the modified LCA are still better than that by the basic version. Therein, the solutions by $P = 10$ are superior to the solutions by $P = 5$. Moreover, Figs. 3 and 4 both demonstrate the modified LCA has faster initial convergence rate. Notably, the objective values by the modified LCA with $P = 10$ as shown in both tables are smaller than 2.937 by IDP.

In the above solutions, the required information for the time-delay state is determined by the DDE solver. Data interpolation is never used. Therefore, the use of the proposed algorithm is very direct and convenient. Because of forward solution manner of LCA, the resultant problem sizes, compared to IDP, are much smaller.

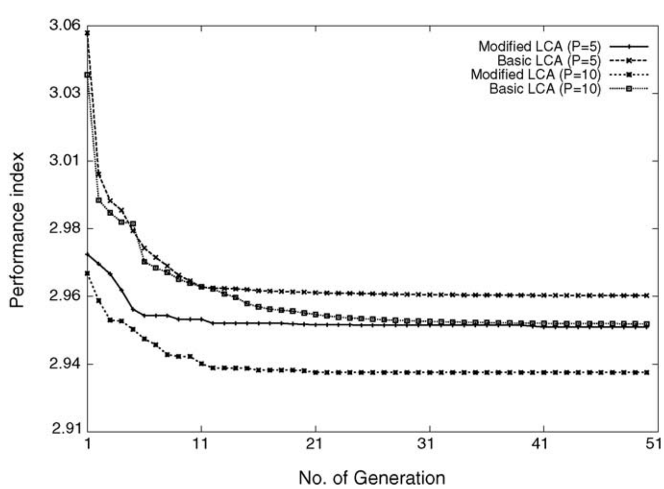

Fig. 3. The convergence curves for the cases of Example 1, where $N_m = 6, N_f = 8$ and $\beta = 0.9$.

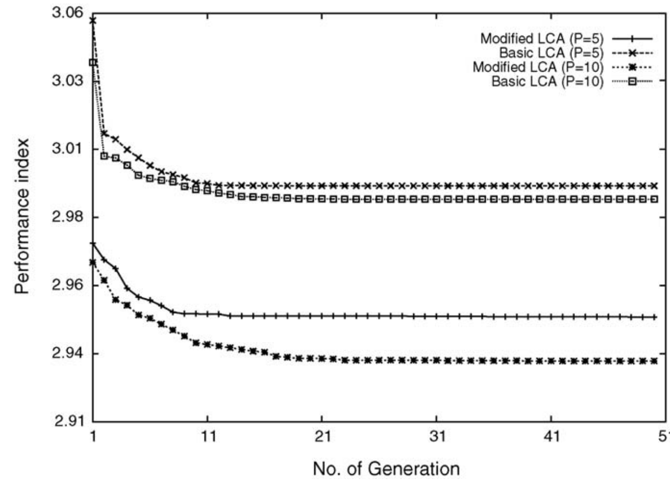

Fig. 4. The convergence curves for the cases of Example 1, where $N_m = 8, N_f = 10$ and $\beta = 0.8$.

Furthermore, in spite of using different N_f and N_m, the modified LCA may acquire the solutions with very similar value, which demonstrates the robustness of LCA to different initial conditions of optimization. Figs. 5 and 6 respectively show the optimal state trajectories and the corresponding input by the modified LCA with $P = 10$, $N_f = 6$ and $N_m = 8$.

Example 2. A harmonic oscillator with retarded damping (Banks and Burns, 1978; Teo *et al.*, 1984).

The model for this example are described by the differential equations

$$\dot{x}_1 = x_2(t)$$
$$\dot{x}_2 = -x_1(t) + x_2(t-1) + u(t)$$
$$\dot{x}_3 = \frac{1}{2}u^2(t)$$

with the initial conditions

$$\mathbf{x} = [10. \quad 0. \quad 0.]'$$

and the initial profile of x_2 is

$$x_2(t) = 10. \quad -1 \le t < 0$$

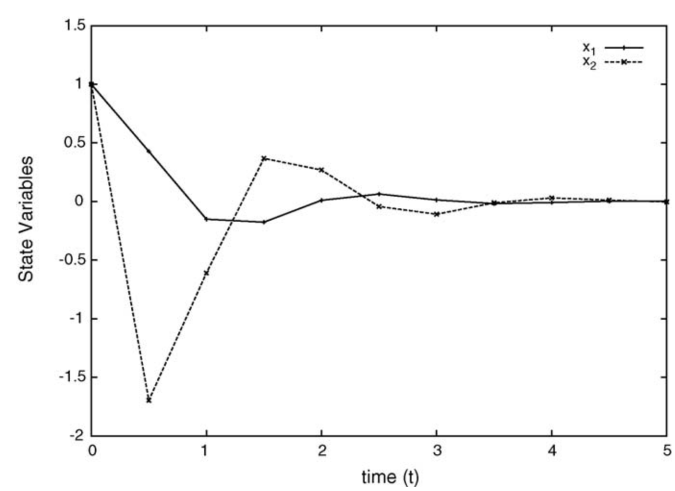

Fig. 5. The state trajectories for Example 1 by the modified LCA with $P = 10$, $N_f = 6$ and $N_m = 8$; $\mathcal{I} = 2.9319$.

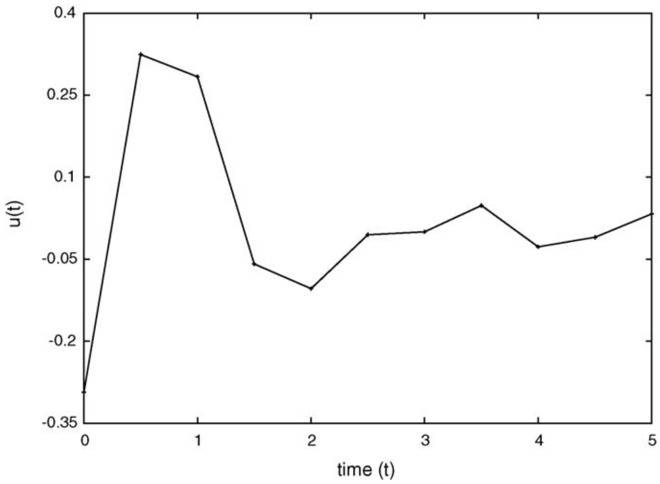

Fig. 6. The control input for Example 1 by the modified LCA with $P = 10$, $N_f = 6$ and $N_m = 8$; $\mathcal{I} = 2.9319$.

The aim of this example is to find the optimal control that may minimize the objective function at $t_f = 2$.

$$\mathcal{I} = 5x_1^2(t_f) + x_3(t_f)$$

Banks and Burns (1978) estimate the approximate objective value to be 3.2587 by solving the corresponding two-point boundary value problem, while Teo *et al.* (1984) reduce the optimal value to 3.4023 and to 3.3991 for the problem with 20 and 48 time divisions.

We initially solve this problem by using 8 time divisions and 200 trial members in one generation. The β factor is set as 0.95. Shown in Table 3, the objective values by the modified LCA are almost the same as the best reported value. The convergence for most solutions by basic LCA are still unsatisfactory. These results further demonstrate the modified LCA is insensitive to the use of different setting. Meanwhile, it is impossible to LCA that the global optimum can be reached by small converted problem. Once the trial members are increased to 300 in one generation, the solutions by basic LCA shown in Table 4 get worse. Even using different N_f and N_m, notably, the modified LCA still converges the solutions to the vicinity the best reported. Fig. 7 shows the optimal input of this example by acquired LCA using $P = 8$, $N_f = 10$ and $N_m = 30$.

Example 3. A modified singular control problem with time delay (Yeo, 1980).

By modifying x_2 into a time-delay state, the third example is composed of the differential equations

Table 3
The objective values for Example (2) by LCA with $\beta = 0.95$.

P	N_f	N_m	Basic	Modified
8	4	50	3.5109	3.4012
	5	40	3.4040	3.3998
	10	20	3.4853	3.3996
	20	10	3.4864	3.3996
10	4	50	3.4554	3.4034
	5	40	3.4637	3.4006
	10	20	3.4997	3.4009
	20	10	3.4772	3.4004

Table 4
The objective values for Example 2 by LCA with $\beta = 0.9$.

P	N_f	N_m	Basic	Modified
8	5	60	3.5541	3.4005
	6	50	3.6295	3.3993
	10	30	3.5467	3.3993
	15	20	3.5641	3.3993
10	5	60	3.5572	3.4010
	6	50	3.6912	3.3998
	10	30	3.5718	3.4000
	15	20	3.4981	3.3998

$$\dot{x}_1 = x_2(t - \tau)$$
$$\dot{x}_2 = -x_3(t)u(t) + 16x_5(t) - 8.$$
$$\dot{x}_3 = u(t)$$
$$\dot{x}_4 = x_1^2(t) + x_2^2(t) + 0.0005(x_2(t) + 16.x_5(t) - 0.1x_3(t)u^2(t) - 8.)^2$$
$$\dot{x}_5 = 1.$$

with the initial conditions

$$\mathbf{x}_0 = [0. \quad -1. \quad -\sqrt{5} \quad 0. \quad 0.]'$$

and the initial profile

$$x_2(t) = -1. \qquad -\tau \le t \le 0$$

The control input is bounded by

$$-4. \le u(t) \le 10.$$

The goal of time problem is to find the input $u(t)$ so as to minimize the objective function at $t_f = 1.0$.

$$\mathcal{I} = x_4(t_f)$$

Notably, the original problem without time delay has been discussed by Yeo (1980) and Luus (1989). The objective function for the delay-free problem are reported to be around 0.1200.

In this example, the values of the time delay τ are changed from 0.1 to 0.5. Using 100 total trial members, we initially assign $N_g = 50$, $N_f = 5$ and $N_m = 20$ to solve this problem. Therein, the β factor is given as 0.9. The corresponding optimal values for $P = 10$ and $P = 20$ are shown in Table 5. Obviously, the modified LCA acquires smaller objective value than the basic LCA, and these

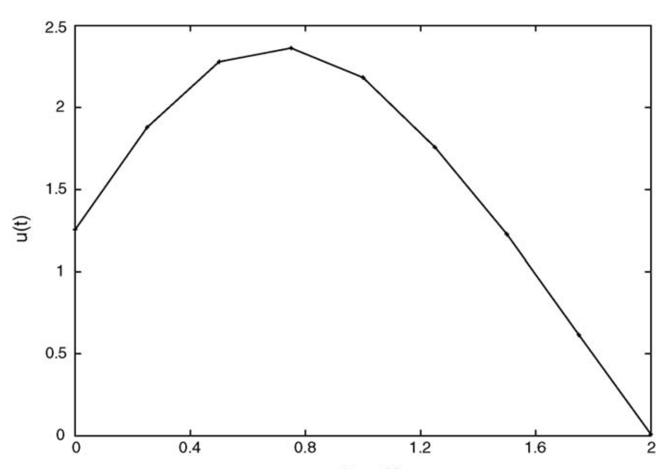

Fig. 7. The optimal input for Example 2 by LCA with $P = 8$, $N_f = 10$ and $N_m = 30$; $\mathcal{I} = 3.3993$.

Table 5
The objective values by LCA with $(N_g, N_m, N_f) = (50, 5, 20)$ for Example 3 with different time delay.

τ	$P = 10$		$P = 20$	
	Basic	Modified	Basic	Modified
0.1	0.20116	0.14526	0.16877	0.14588
0.2	0.23027	0.17825	0.20409	0.17782
0.3	0.25693	0.21354	0.23914	0.21423
0.4	0.30693	0.25251	0.27784	0.25298
0.5	0.33951	0.29396	0.31808	0.29470

Table 6
The objective values obtained by LCA with $(N_g, N_m, N_f) = (50, 10, 10)$ for Example 3. with different time delay.

τ	$P = 10$		$P = 20$	
	Basic	Modified	Basic	Modified
0.1	0.20646	0.14580	0.19461	0.14590
0.2	0.24203	0.17789	0.23969	0.17951
0.3	0.28322	0.21403	0.27812	0.21462
0.4	0.34077	0.25271	0.31336	0.25491
0.5	0.37017	0.29417	0.34916	0.29584

results are also very close each other. Once N_f and N_m are both changed to 10, the objective values by the modified LCA in Table 6 have no significant changes compared to the results in Table 5. However, the solutions by the basic LCA have large fluctuations. It seems to imply that the modified LCA is more robust to the initial conditions of optimization than the basic LCA. For the cases with $P = 10$, $N_f = 10$ and $N_m = 10$, the optimal input profiles for the problems with different time delay are shown in Fig. 8.

Example 4. A time-varying time-delayed system (Dabedo and Luus, 1992; Luus, 2000; Palanisamy *et al.*, 1988).

The fourth problem is described by the differential equations

$$\dot{x}_1 = t x_1(t) + x_1(t-1) + u(t)$$
$$\dot{x}_2 = x_1^2(t) + u^2(t)$$

with the initial conditions

$$\mathbf{x}_0 = [1. \quad 0.]'$$

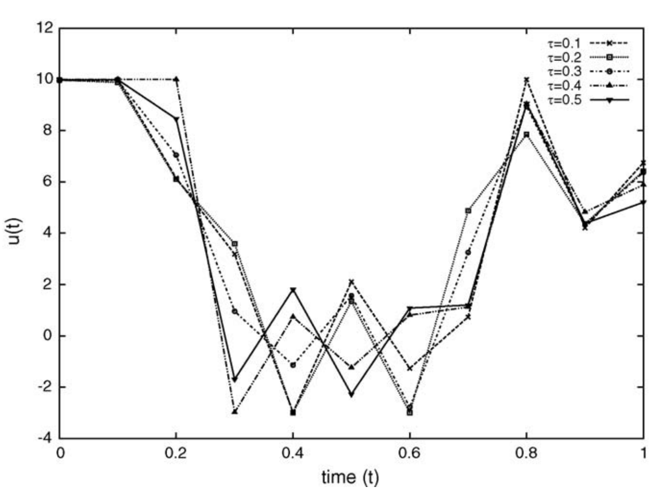

Fig. 8. The control inputs for Example 3 with different time delay by LCA with $P = 10$, $N_f = 5$ and $N_m = 6$.

Table 7
The objective values for Example 4 by LCA with $P = 4$ and $\beta = 0.9$.

P	N_f	N_m	Basic	Modified
4	4	50	5.0611	4.7976
	5	40	5.2022	4.7970
	10	20	5.1847	4.7973
	20	10	5.1790	4.7970

Table 8
The objective values for Example 4 by LCA with $P = 6$ and $\beta = 0.85$.

P	N_f	N_m	Basic	Modified
6	4	50	5.1811	4.7977
	5	40	5.5198	4.8004
	10	20	5.0687	4.7970
	20	10	5.2378	4.7976

and the initial profile

$$x_1(t) = 1.0 \quad -1 \le t < 0.$$

The aim of this problem is to minimize $x_2(t_f)$ at $t_f = 2.0$.

$$\mathcal{I} = x_2(2)$$

By the use of single term Walsh series, Palanisamy *et al.* (1988) report the minimum value to be 6.0079. Using iterative dynamic programming with 20 fixed divisions, Dadebo and Luus (1992) determine the minimum value to be 5.0674. This value is largely reduced to 4.7968 by using RIFTI and 10 variable time divisions (Luus, 2000). However, the computational efforts including ten-phase executions, each consisting 10 iterations, are still very formidable.

To acquire a satisfactory solution, the total trial members in each generation are increase to 200 in this example. After 50 generations, the solutions obtained by both versions of LCA for $P = 4$ are shown in Table 7. In these cases, β is set as 0.9. The objective values by the modified LCA may approach to the best reported value within 0.1%. Once P is increased to 6 and β is reduced to 0.85, The values shown in Table 8 are still very close to the above results. With respect to IDP, the problem sizes used by LCA are small. Thus, the computational efforts of LCA can be largely saved. The trajectories of the optimal inputs by modified LCA with $P = 4$ and $P = 6$ are shown in Fig. 9.

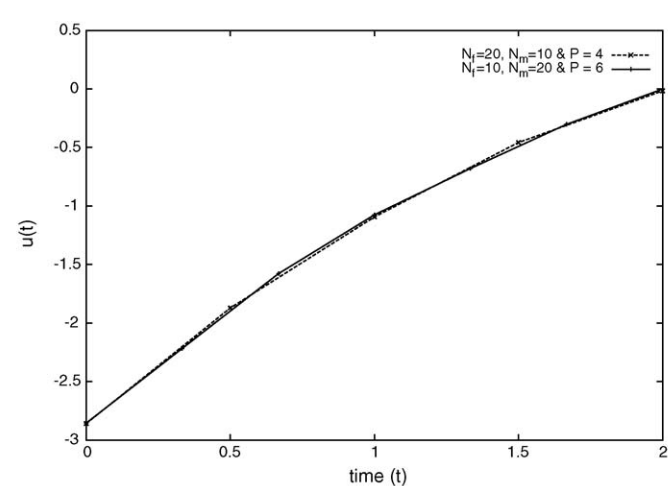

Fig. 9. The optimal inputs for Example 4 by LCA with $P = 4$ and $P = 6$.

Table 9
The objective values and terminal errors of x_1 for Example 5 by LCA with $\beta = 0.9$ and $\omega = 1.0$.

P	N_f	N_m	$\mathcal{I}$	$x_1(2)$
2	5	50	0.09369	-1.7354×10^{-4}
	10	25	0.09373	-3.1060×10^{-4}
	25	10	0.09392	-3.9971×10^{-5}
4	5	50	0.09390	-4.9633×10^{-4}
	10	25	0.09429	-8.9099×10^{-5}
	25	10	0.09389	-2.6988×10^{-5}
6	5	50	0.09417	-2.2609×10^{-3}
	10	25	0.09460	-8.1096×10^{-5}
	25	10	0.09493	-7.6992×10^{-4}

Example 5. The minimum energy control problem with terminal constraints (Chyung and Lee, 1966; Dadebo and McAuley, 1995).

The fourth problem to be discussed is described as

$$\dot{x}_1 = -x_1(t-1) + u(t)$$
$$\dot{x}_2 = \frac{1}{2}u^2(t)$$

with the initial conditions

$$\mathbf{x}_0 = [1. \quad 0.]'$$

and the initial profiles

$$x_1(t) = 1.0 \quad -1. \le t \le 0.$$

The control input is also restricted by

$$u(t) \ge 0.$$

Thus, the aim of this problem is to find the optimal input so that the objective function is minimized

$$\mathcal{I} = x_2(t_f)$$

and $x_1(t)$ is driven to the origin at $t_f = 2.0$, namely $x_1(2) = 0$.

Chyung and Lee (1966) analytically determine the objective value to be 0.09375. To handle the terminal constraint, Dadebo and McAuley (1995) recommend using the weighted-absolute-error function to penalize the violation. Thus, the objective function is augmented as

$$\mathcal{I} + \sum_{\ell=1}^{\eta} \xi_\ell |x_\ell(t_f) - x_\ell^\star| \quad 1 \le \ell \le n \tag{20}$$

where η denotes the numbers of state variable with terminal constraint. ξ_ℓ is the weighting factor corresponding to ℓ-th state variable, x_ℓ, restricted by the terminal value $x_\ell^\star$. Additionally, they also suggest using quadratic approximation to improve the estimation of the time-delay state. As a result, the objective value is converged to 0.094963 by using IDP with 21 state grids, 5 trial inputs and 8 time divisions. The terminal error is converged to -3.2774×10^{-4}. Once the state grids are further added to 41, the objective value is refined to 0.093538 and the terminal errors of $x_1(2)$ is -7.464×10^{-3}. Similar to Example (4), the above computational efforts are very large owing to the enlarged problem size.

To acquire satisfactory solutions, we set the total trial members in one generation as 250 in this example. To make a fair comparison, we use the same augmented objective function as Eq. (20) to handle the terminal constraint. The penalty factor in Eq. (20) is given as 1.0. The time divisions are respectively set as 2, 4 and 6. Table 9 shows these results by the modified LCA with $\beta = 0.9$. Obviously, most objective values are converged to the analytical solution within 0.1%. LCA further uses very small problem size to approach the global solution. Additionally, even a very wider range of penalty factor, such as $\omega = 100$ is used, the objective function is still converged to the analytical solutions within 0.1%. The terminal errors may be kept lower than 1.0×10^{-3}.

Table 10
The comparisons of the objective values for Example 6 by IDP and LCA with $P = 8$ and $N_g = 100$.

τ	N_f	N_m	$\mathcal{I}$	$\mathcal{I}^*$
0.2	5	20	0.023699	0.023731
	10	10	0.023727	
	20	5	0.023713	
0.4	5	20	0.024480	0.024615
	10	10	0.024498	
	20	5	0.024506	
0.6	5	20	0.025092	0.025192
	10	10	0.025115	
	20	5	0.025114	
0.8	5	20	0.025514	0.025499
	10	10	0.025548	
	20	5	0.025639	

Example 6. Nonlinear two-stage continuous stirred tank reactor system (Luus *et al.*, 1995).

Originally developed by Aris and Amundson (1958) and further used by Lapidus and Luus (1967) for stability analysis, this problem consists of four time-delay differential equations described by

$$\dot{x}_1 = 0.5 - x_1(t) - R_1 = f_1(t)$$
$$\dot{x}_2 = -2[x_2(t) + 0.25] - u_1[x_2(t) + 0.25] + R_1 = f_2(t)$$
$$\dot{x}_3 = x_1(t-\tau) - x_3(t) - R_2 + 0.25$$
$$\dot{x}_4 = x_2(t-\tau) - 2x_4(t) - u_2[x(9t) + 0.25] + R_2 - 0.25$$

where x_1 and x_3 are normalized concentration variables in tanks 1 and 2; x_2 and x_4 are normalized temperature variables in tanks 1 and 2. R_1 and R_2 denote the reaction terms in tanks 1 and 2, which are described as

$$R_1 = [x_1(t) + 0.5]\exp\left(\frac{25x_2(t)}{x_2(t) + 2}\right)$$

$$R_2 = [x_3(t) + 0.5]\exp\left(\frac{25x_4(t)}{x_4(t) + 2}\right)$$

The initial state profiles for x_1 and x_2 and the initial conditions for x_3 and x_4 are

$$x_1(t) = 0.15 \quad -\tau \le t \le 0$$
$$x_2(t) = -0.03 \quad -\tau \le t \le 0$$
$$x_3(0) = 0.1$$
$$x_4(0) = 0.$$

Table 11
The best objective values for Example 6 with $\tau = 0.8$ by LCA with $P = 8$ and $N_g = 100$.

N_f	N_m	$\mathcal{I}$
5	20	0.025496
10	10	0.025465
20	5	0.025493

The normalized controls are restricted by

$$-1 \le u_j \le 1 \quad j = 1, 2$$

This problem is to the controls u_1 and u_2 in the time duration from 0 to 2 to minimize the objective function

$$\mathcal{I} = \int_0^2 [x_1^2 + x_2^2 + x_3^2 + x_4^2 + 0.1(u_1^2 + u_2^2)]\mathrm{d}t$$

As stated in Section 1, the evaluation of the objective function can be facilitated by adding a new state variable x_5 by

$$\dot{x}_5 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 0.1(u_1^2 + u_2^2)$$

with the initial condition $x_5(0) = 0$.

For this problem, many preliminary tests show that the convergence to the optimal solution could not be obtained from an arbitrary chosen initial policies. Such convergent difficulty also encountered in the solution by IDP (Luus *et al.*, 1995) might be derived from the nature of discontinuity and high nonlinearity. In order to acquire an feasible initial input, we first use the Taylor's expansion to approximate the time-delay term as follows

$$x(t - \tau) \approx x(t) - \tau \frac{\mathrm{d}x}{\mathrm{d}t_1}$$

As a result, the original model is reposed as a delay-free problem

$$\begin{aligned}
\dot{x}_1 &= f_1(t) \\
\dot{x}_2 &= f_2(t) \\
\dot{x}_3 &= x_1(t) - x_3(t) - \tau f_1 - R_2 + 0.25 \\
\dot{x}_4 &= x_2(t) - 2x_4(t) - u_2[x_4(t) + 0.25] - \tau f_2 + R_2 - 0.25
\end{aligned}$$

Once the value of τ is given as 0.2, a rough but feasible control policy can be determined by LCA with $P = 8$, $N_g = 10$, $N_f = 10$ and $N_m = 100$, and the corresponding objective value is converged to 0.02486. Using the above feasible policy as initial guess and giving a range of ± 0.003, LCA re-solves the original model with $\tau = 0.2$. The acquired minimum values with various N_f and N_m can be seen in Table 10. Therein, the values shown below $\mathcal{I}^*$ are the results by IDP. LCA may get more smaller objective value than IDP for the case with $\tau = 0.2$. We further use the previous feasible input as the initial guess of LCA to determine the optimal controls for the cases with $\tau = 0.4, 0.6$ and 0.8. The corresponding objective values are shown in Table 10. Additionally, the results by IDP are also list in the same table. In the case of $\tau = 0.4$ and $\tau = 0.6$, LCA has better resultant convergence than IDP. For the case of $\tau = 0.8$ the objective value by LCA are slightly larger than the reported one by IDP. To promote the numerical quality, we use the optimal input acquired from the case with $\tau = 0.6$ as the initial guess to re-solve the case with $\tau = 0.8$. As shown in Table 11, LCA with various N_f and N_m may further refine the objective value toward the known optimum. The optimal inputs for different time delay are shown in Fig. 10.

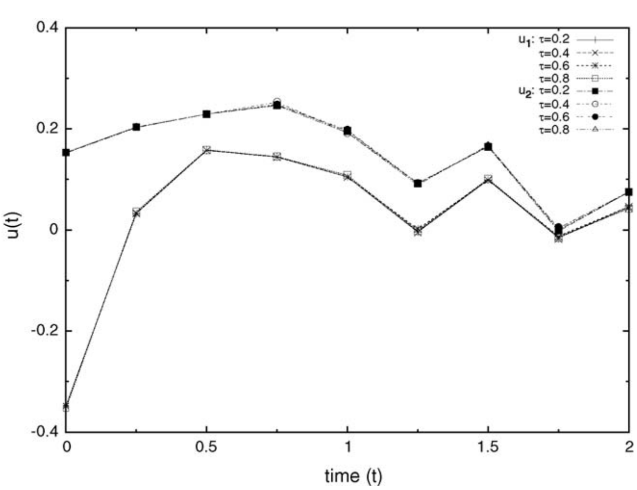

Fig. 10. The optimal inputs for Example (6) with various time delay by LCA.

6. Conclusion

In this study, the line-up competition algorithm coupled with the delay differential equation solver is applied to solve time-delay optimal control problems. Based on the concept of control vector parametrization, The TDOCP is first discretized, and then optimized by the proposed algorithm. In order not to increase the discontinuity, the continuous input in this study is approximated by a series of ramp inputs with fixed time divisions. Owing to forward solution manner, the proposed algorithm barely use the estimation of time-delay states in most problems. Thus, the solution framework becomes very convenient and straightforward. To efficiently generate new trial members, the uniform sampling policy is replaced by the normal sampling strategy. Six typical problems are used to demonstrate the performance of the proposed method. From the numerical illustrations, the modified LCA is proved to be efficient in convergence and robust to the initial conditions of the optimization. Furthermore, because the method may use small discretized problems to converge the solution, the computational loads are also largely reduced.

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