Journal of Mathematical Behavior 29 (2010) 86–98

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Are beliefs believable? An investigation of college students' epistemological beliefs and behavior in mathematics

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ARTICLE INFO

Keywords:
Epistemological belief
Calculus
Problem solving behavior
Metacognition

ABSTRACT

College students' epistemological belief in their academic performance of mathematics has been documented and is receiving increased attention. However, to what extent and in what ways problem solvers' beliefs about the nature of mathematical knowledge and thinking impact their performances and behavior is not clear and deserves further investigation. The present study investigated how Taiwanese college students espousing unlike epistemological beliefs in mathematics performed differently within different contexts, and in what contexts these college students' epistemological beliefs were consistent with their performances and behavior. Results yielded from the survey of students' performances on standardized tests, semi-open problems, and their behaviors on pattern-finding tasks, suggest mixed consequences. It appears that beliefs played a more reliable role within the well-structured context but lost its credibility in non-standardized tasks.

1. Introduction

The role of college students' epistemological beliefs (i.e., beliefs about the nature of knowledge and knowing) in their academic performance of mathematics has been documented and is receiving increased attention. In general, studies of epistemological beliefs at the college level can be traced back to Perry's (1968) survey of undergraduates' intellectual development. He found that college freshmen typically espouse a belief that knowledge is simple, certain, and assorted in canonical form. Schoenfeld (1983) explored the influence of college students' epistemological beliefs about mathematics on their problem solving ability. Students' non-cognitive behavior led Schoenfeld to establish an evidence-based framework of mathematical problem solving in which belief systems play the most fundamental and subtle role (Schoenfeld, 1985). In terms of Schoenfeld, purely cognitive behavior is rare and individuals' perspectives regarding the nature of tasks may affect their intellectual performance within a particular context. Namely, one's mathematical worldview shapes the way one does mathematics (Schoenfeld, 1985, 1992). Schoenfeld's claim has been generally endorsed by several studies conducted during the 1990s and early 2000s (Carlson, 1999; Higgins, 1997; Kloosterman & Stage, 1991; Presmeg, 2002). However, to what extent and in what ways problem solvers' beliefs about the nature of mathematical knowledge and thinking impact their performances and behavior is not clear and deserves further investigation. The present study attempted to delve into the issue through investigating Taiwanese college students' epistemological beliefs about mathematics and analyzing multiple facets of their problem solving performances and behavior.

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0732-3123/$ – see front matter
doi:10.1016/j.jmathb.2010.05.001

2. Theoretical background

The early literature had indicated that, in the search for knowledge, we do not start from a neutral zero and the human mind "is a belief-seeking rather than a fact-seeking apparatus" (Jastrow, 1927, p. 284). Belief is seen as an individual's psychological state regarding the truth of particular propositions or personal premise for specific objects. It is the result of reflecting on actions that may or may not be knowledge-based and intellectually verified. Following two decades of studies on mathematical problem solving during the 1980s and 1990s, it has been generally assumed that individuals' mathematics-related beliefs may have a significant impact on their mathematical thinking and behavior. Nonetheless, owing to its complicated and contextualized nature, a widely accepted definition of belief is lacking and the discourse on how it works is diverse. The loose use of definition and varied interpretations may explain why the role of beliefs in mathematics education remains peripheral and hidden (Leder, Pehkonen, & Törner, 2002; MacLeod, 1992). Apparently, "research on this topic has not yet resulted in a comprehensive model of, or theory on, students' mathematics-related beliefs," and "the state of the art of the research field does not allow the development of a comprehensive theory at the moment" (Op't Eynde, De Corte, & Verschaffel, 2002, p. 15).

Rokeach (1968) defined belief as "any simple proposition, conscious or unconscious, inferred from what a person says or does" (p. 113), and distinguished its content into descriptive belief, evaluative belief, and prescriptive belief, which are different but interrelated components (Fig. 1(a)). For instance, "I believe students learn something useful in school," is descriptive and "I believe mathematics is useful," is evaluative. A mixture of the two beliefs may entail "I believe students should learn mathematics," which is prescriptive. Törner (2002) structured mathematical beliefs in a hierarchical form in which global beliefs (general beliefs of teaching and learning), domain-specific beliefs (beliefs about specific domains such as algebra or geometry), and subject-matter beliefs (beliefs of amount and organization of the subject) interact among each other via top-down or bottom-up influences (Fig. 1(b)). In terms of Rokeach and Törner, "I believe students should learn more algebra than geometry in school because algebra is more useful than geometry," is categorized as a domain-specific evaluative belief. Taking its multi-faceted natures into account, focusing on a certain particular belief rather than exploring the whole belief system may be more feasible and rewarding (Pajares, 1992).

Epistemological beliefs refer to beliefs about the nature of knowledge and knowing. During the past decade, the study of epistemological beliefs has increasingly received attention among educational psychologists. Relative findings suggest epistemological beliefs are linked to students' academic performance in a distinct context (Bendixen & Hartley, 2003; Kardash & Howell, 2000; Schommer-Aikins, Duell, & Hutter, 2005; Whitemire, 2004). While exploring the relationship between undergraduates' epistemological beliefs and information-seeking behavior in a digital environment, Whitemire (2004) found students at higher stages of epistemological development displayed better ability to evaluate information sources and to deal with conflicting situations. Bendixen and Hartley (2003) not only claimed that pre-service teachers' epistemological beliefs were significantly related to their achievement in a hypermedia learning course, but also suggested that epistemological beliefs were more linked to ill-defined problem solving. Furthermore, a quantitative survey conducted by Kardash and Howell (2000) indicated college students viewing the process of learning as clear-cut and unambiguous tended to believe that memorization plays a major role in learning and knowledge can be known with certainty.

Schoenfeld (1983) had performed the pioneering work in the specific domain of mathematics. Through investigating how two college freshmen approached geometrical construction tasks, he asserted that purely cognitive problem solving behavior was rare and individuals' belief systems of mathematics played a key role behind the scenes. His succeeding surveys (Schoenfeld, 1988, 1989) not only endorsed his previous claim, but also further suggested that students' mathematical behaviors seem to be driven by earlier experiences that shape their subsequent beliefs. Kloosterman and Stage (1991) also found low college mathematics achievers generally had a poor conception of the nature of mathematics, and their beliefs about mathematics were related to the final course grade. By employing VAMS (Views About Mathematics Survey), Carlson (1999) found mathematics graduate students held expert-like views, while pre-calculus students held common views. Furthermore, by observing the graduate students' problem solving behavior while completing complex mathematical tasks, she suggested that non-cognitive factors play a prominent role in a student's mathematical success. Based upon evidence yielded from two research projects, Presmeg (2002) was convinced that either high school or graduate students' beliefs about the nature of mathematics enable or constrain their capability to bridge the links between everyday practice and mathematical concepts taught in school. While studying the relationship between middle school students' epistemological beliefs and aca-

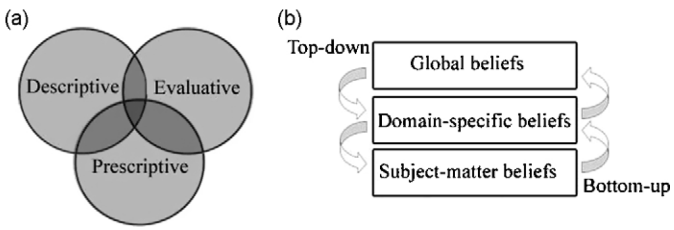


Fig. 1. (a) Different belief components and (b) different belief structures.

demic performance, Schommer-Aikins et al. (2005) proposed that both general epistemological beliefs and domain-specific mathematical problem solving beliefs could predict students' academic performance and grade point average.

In spite of the aforementioned studies consistently suggesting the effects of epistemological beliefs on students' performance or achievements, some major issues may have been neglected. Epistemological beliefs mostly have been studied quantitatively in four aspects: simplicity of knowledge, certainty of knowledge, source of knowledge, and justification of knowledge. It is believed that a qualitative approach may be complementary to a quantitative survey and will shed more light on how beliefs interact with cognitive behavior. In particular, several studies have noted the significant role of metacognition, which is fundamentally tied to beliefs (Bendixen & Hartley, 2003; DeFranco, 1996; Garofalo & Lester, 1985; Lester, Garofalo, & Kroll, 1989; Schoenfeld, 1985, 1987). For instance, while observing two groups of practicing mathematicians' problem solving behavior, DeFranco (1996) found an unanticipated result in that, contrary to nationally known mathematicians, those mathematicians who earned a doctorate degree in mathematics but have not achieved recognition may perform like novices. An analysis suggests the two groups of mathematicians differed in strategic, meta-cognitive, and belief aspects. Metacognition refers to one's knowledge concerning one's own cognitive processes or anything related to them (Flavell, 1976) and "has two separate but related aspects: (a) *knowledge* and *beliefs* about cognitive phenomena, and (b) the *regulation* and *control* of cognitive actions" (Garofalo & Lester, 1985, p. 163). It is higher order thinking that involves control ability over the thinking processes that is better to be investigated qualitatively. In a simpler sense, it can be interpreted as "knowing about knowing" (Metcalfe & Shimamura, 1994) or "thinking about thinking" (Butler & Winne, 1995). Bendixen and Hartley (2003) indicated students' meta-cognitive awareness will significantly mediate success in most learning environments. Yet meta-cognitive awareness is a missing component in Bendixen and Rule's (2004) model for epistemological development because achieving more sophisticated beliefs would not be possible if meta-cognitive awareness never occurs during the learning process (Liu, 2009). However, the effect of epistemological beliefs and meta-cognitive awareness on college students' mathematical behavior and performance have rarely been documented in a qualitative way. By taking this concern into account, the present study investigated the following issues:

(1) To what degree did college students espousing unlike epistemological beliefs in mathematics perform differently on standardized items?
(2) To what extent did college students espousing unlike epistemological beliefs in mathematics approach semi-open homework problems differently?
(3) In what ways did college students espousing unlike epistemological beliefs in mathematics behave differently in pattern-finding tasks?
(4) In what contexts were college students' epistemological beliefs consistent with their performance and behavior?

On the basis of these findings, the present study attempted to reveal potential variables hidden behind interrelationships among personal epistemology, metacognition, mathematical performance, and problem solving behavior.

3. Data sources and research methods

The bulk of an individual's beliefs, much like an iceberg, is beneath the surface and is not directly accessible (Buehl & Alexander, 2001). If beliefs are hidden, how can we demonstrate that they are present (Mason, 2004)? Furthermore, one's mathematical behavior or performance may not be identified by any single task or instrument. Taking these multidimensional features of epistemological beliefs and behaviors into account, five data sources were developed to serve the purpose of this observational study: (1) the epistemological belief questionnaire; (2) individual interviews; (3) standardized calculus problems; (4) semi-open problems; (5) pattern-finding problems. The first two instruments were employed to identify students' epistemological beliefs about mathematics; the latter three were used to depict their mathematical behaviors and performances.

3.1. Qualitative epistemological belief questionnaire

The main focus of the qualitative epistemological belief questionnaire is the individual's beliefs of the nature of mathematical thinking and mathematical knowledge. The questionnaire had been developed via 4-stage pilot tests and employed in several studies (Liu & Niess, 2006; Liu, 2007, 2009). An intra-triangulation measurement, involving different but relative items, was designed to detect students' inner perceptions of a specific construct. For example, for probing their beliefs of mathematical thinking, respondents were asked to define the term directly (*In your understanding, what is mathematical thinking*?) and express their views of mathematicians' ways of thinking (*In your understanding and imagination, how do mathematicians think while solving a problem*? *Is there any difference between a mathematician's way of thinking and a layperson's*?), and the role of creativity in mathematical thinking (*Some hold that solving mathematical problems is a thinking activity involving creativity; others argue that getting correct answers requires following predetermined, known procedure, what is your opinion about this*?). For understanding their beliefs of mathematical knowledge, respondents were asked to give a direct definition (*In your opinion, what is mathematics*?), compare the essence of mathematics and other disciplines (*What makes mathematics different from other disciplines? What are the major similarities or differences between mathematics and science/art*?), and react to the "invented vs. discovered" debated issue (*Some claim that mathematical knowledge is a discovered truth, but others* *consider it as a man-made product invented by mathematicians, what is your opinion about this*?), both supplementary to their characterization of mathematics.

At the first class meeting, 52 students answered the questionnaire and on the basis of their written responses, eight of them judged to espouse relatively extreme beliefs (a sophisticated and dynamic belief on the one end and a naïve and static belief on the other) were invited to join subsequent in-depth investigations.

3.2. Individual interview

In order to achieve a greater degree of credibility, the eight students responded to the identical questionnaire at the intermediate and the end of the academic year, respectively. After responding to the epistemological belief questionnaire, the eight selected students participated in follow-up individual interviews purposing to validate their statements (if they did understand the questionnaire items) and check the reliability of their professed claims (if both written and oral responses were consistent). For instance, if interviewees declared that mathematical thinking involved creativity, the interviewer asked them to give the sources of their answers and examples for supporting their clams. It was found that some interviewees referred mathematical creativity to the flexible use of different formulas to solve problems, rather than relating it to innovative ideas.

In addition to identifying any differences among their accounts given at various times and detecting their core beliefs about mathematics to a satisfied level, the three semi-structured interviews were additionally used to ascertain the eligibility of their participation. Though all eight selected students took part in all activities, after a yearlong observation, four students were excluded from this report for their unstable and ambiguous accounts. Consequently, only four showing relatively stable beliefs became the final targets. Note that one's beliefs may not be knowable without long term investigation and appropriate instruments. Despite the effort made in the present study, owing to the difficulty caused by the limitation of questionnaires and interviews for indentifying individuals' epistemological beliefs, any information yielded may not fully represent participants' holistic worldviews of mathematics.

3.3. Standardized calculus problems

Standardized problems were those items used in the two midterm and two final examinations, mostly collected from the exercises in calculus textbooks. There were 12–15 standardized problems for each examination. They were typically in a traditional and closed form that can be solved by directly applying definitions and formulas or implementing routine procedures. Some exemplary problems are shown in Table 1.

3.4. Semi-open problems

All students in this calculus class were assigned several semi-open problems to help them develop higher order thinking. Solving these tasks not only required integrating fundamental content knowledge learned from calculus, but also demanded breakthrough creative thinking. For probing students' performance within various contexts, several semi-open problems were given as midterm or final examination items. Other tasks were assigned as homework, allowing 1 or 2 weeks of working time. Table 1 displays some exemplary semi-open problems.

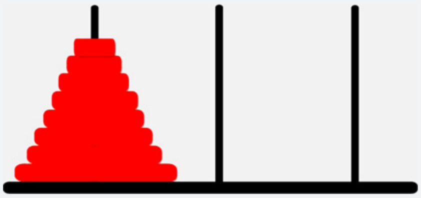
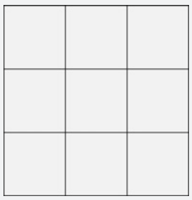
3.5. Pattern-finding problems

Both standardized and semi-open problems were intimately connected to the calculus curriculum and may be heavily knowledge-laden. Furthermore, if on-site observation were lacking, it would be difficult to know how the students approached these problems and what strategies were involved during their problem solving processes. Therefore, the eight initially selected students were asked to solve three pattern-finding problems right after each of the three follow-up interviews. Three pattern-finding problems (Table 1) had nothing to do with the calculus curriculum and required only basic knowledge. No time restriction was set up to finish and students were freely able to decide when they would stop working on the task. Tower of Hanoi-1 problem asked students to shift all eight rings from one pole to another; neither minimizing nor counting their moves was required. Tower of Hanoi-2 problem, however, asked the students to decide the minimal move for shifting all eight rings from one pole to another. The third problem, magic square problem, required students to place the numbers 1 through 9 in a 3 by 3 grid to make the sum of the numbers in each row, column, and diagonal all equal. The whole process of students' problem solving behaviors on these tasks were videotaped, transcribed, and annotated.

3.6. Framework for analyzing epistemological beliefs

Relative studies have proposed four general dimensions for investigating epistemological beliefs, including simplicity of knowledge, certainty of knowledge, source of knowledge, and justification of knowledge (Hofer & Pintrich, 1997; Pintrich, 2002). On the basis of this framework, the present qualitative investigation probed personal epistemology in terms of two different but intertwined facets: (a) beliefs about mathematics as a process, micro-conceptions regarding mathematical practice, and (b) beliefs about mathematics as a product, macro-conceptions regarding mathematical facts. The former refers to students' synthetic views about how rules and creativity are involved in mathematical thinking, dynamicity of

Table 1
Exemplary standardized, semi-open, and pattern-finding problems.

Type of problems	Statement of the problems
Standardized	(1) Evaluate $\lim_{x\to 1}\frac{x^2+4}{x-1}$; (2) $f(x)=\begin{cases} x^2\sin\frac{1}{x}, x\neq 0 \\ 0, x=0\end{cases}$ derive $f'(0)$ by using definition; (3) determine the convergence of $\sum_{k=1}^{\infty}\frac{4}{k(k+2)}$.
Semi-open	*Fly problem*: Two trains are traveling toward each other on the same track, each at 60 km/h. When they are exactly 120 km apart, a fly takes off from the front of one of the trains, flying toward the other train at a constant rate of 100 km/h. When the fly reaches the other train, it instantly changes directions and starts flying toward the other train, still at 100 km/h. It keeps doing this back and forth until the trains finally collide. How much total distance has the fly travelled when the trains finally collide? *Salesman problem*: On Monday morning, a salesman leaves home at 7:13 A.M. and arrives at his office at 8:58 A.M. On Tuesday morning, he leaves home at 7:16 A.M. and arrives at his office at 8:56. The salesman notices that a nearby train station clock points 8:00 on both days. Therefore, he must have been at the same location at the same time on both days. But his wife does not believe such a coincidence could occur. Is the salesman's claim true or false? Please defend your answer.
Pattern-finding	*Tower of Hanoi-1*: Please transfer the entire 8 discs to another rod by obeying the rules that moving only one disc at a time and never a larger one onto a smaller.  *Magic square*: Please arrange the numbers 1–9 in the following 3 × 3 grid such that the sum of the three numbers in any horizontal, vertical, or main diagonal line is always the same. 

mathematical thinking, and mathematicians' strategies in mathematical thinking. The latter represents students' synthetic views about how mathematical knowledge originates and progresses, the factors and conditions involved in the expansion of mathematical knowledge, and certainty of mathematical knowledge.

3.7. Framework for analyzing performance and behaviors

By referring to relevant studies, several frameworks have been employed to fit the purpose of individual study. Glass (2002) adopted a coding scheme consisting "code of strategies," "code of justification," and "code of connection made," to analyze college students' behavior on non-routine problems. Protocol analysis was used by Cifarelli and Cai (2005) and Lerch (2004) to probe college students' use of strategies and control behavior. Taking the goal and the nature of tasks used in the present study into account, participants' performance and behaviors were rated on the qualitative as well as quantitative basis, along with a protocol analysis. Particular attention was paid to in what way and to what extent these problem solvers elicited learned knowledge, developed effective strategies, and controlled their behaviors while working on the problems.

A three-level index was applied to assess students' performance on the standardized problems. It was graded as H (high performance) if they scored 80% or more; grade M (moderate performance) was for those scoring between 60% and 80%; and L (low performance) represented the score below 60%. A four-aspect rating scheme (engagement, knowledge, strategy, and completeness) combined with the aforementioned three-level index was used to assess students' performance on semi-open problems. Table 2 shows details regarding the four aspects of the rating scheme.

Students' problem solving behaviors on pattern-finding problems were videotaped, transcribed, annotated, and evaluated on a qualitative basis, instead of the three-level index. Because problem solvers' motivation, the use of approaches, the status of struggling, and the accuracy of solutions were also major concerns, yet knowledge played a minor role in pattern-finding problems, the scheme employed to analyze problem solving behaviors shared three common categories (engagement, strat-

Table 2
Rating scheme for analyzing performance on the semi-open problems.

Aspects of performance	Content
Engagement	Degree of problem solvers' involvement.
Knowledge	Knowledge sources that problem solvers used.
Strategy	Approaches that problem solvers adopted to attack the tasks.
Completeness	Extent of the problem solved.

Table 3
Scheme for observing behavior on the pattern-finding problems.

Aspects of behavior	Content
Engagement	Degree of problem solvers' involvement in the activity.
Strategy	Approaches that problem solvers adopted to attack the tasks.
Self-regulation	Status of problem solvers monitoring their own thinking.
Completeness	Extent of the problem solved.

egy, and completeness) with the semi-open problem framework and 'knowledge' was replaced by 'self-regulation' (Table 3). The main focus was on their strategic plans and how they monitored their own thinking to keep the whole activity on track.

4. Results

4.1. Students' epistemological beliefs

Among the eight initially selected students, only four of them (Chun-Pei, Hwa-Chan, Shen-Wei, and Win-Pei, all names are made-up) were considered to be appropriate for the purpose of the present study because of their stable and consistent epistemological beliefs about mathematics across three interviews. Data yielded from the other four students demonstrating vague and unsteady views was excluded. Among the four final participants, two male students (Chun-Pei and Hwa-Chan) expressed a sophisticated belief that mathematical thinking is a logic-based creative and dialectical process involving guessing, testing, proving, and generalizing. In the first and second interview, Hwa-Chan used Rubik's cube as an example of his conception of mathematical thinking:

> Mathematical thinking is a means of seeing things in many ways, like playing Rubik's cube. Resolving Rubik's cube requires us to try various approaches, which is similar to mathematics. A math problem may not be solved by a single method. Instead, it can be attacked from different approaches and still some solutions have not been identified. (First Interview, Hwa-Chan)

Bearing such a dynamic view in mind, Hwa-Chan considered that personal creativity should be more involved in problem solving as opposed to following routine processes.

Both Chun-Pei and Hwa-Chan held that mathematical knowledge grows and expands along with human demands, which could be fallible, as manifested in Chun-Pei's claim:

> In my point of view, [mathematics] is a discovered truth explained by invented knowledge. Something out there could not be changed. [We] don't know how it works and then [we] try to solve it. Methods are invented and imagined by human beings. . .Some errors in mathematics might have not appeared. It could be overthrown by new invented methods. (Chun-Pei, Third Interview)

However, they both believed mathematical facts are absolute truth because the fallibility will be fixed through mathematicians' continuous intelligent effort until certainty is achieved. Furthermore, they appreciated the value of abstract mathematics by acknowledging that mathematicians may explore mathematics for the sake of mathematics itself.

On the other hand, Shen-Wei and Win-Pei (the only female participant) viewed mathematical thinking as an activity of solving arithmetical problems via implementing numbers and formulas. Though they recognized the significant role of logic in mathematical thinking, they interpreted it more as the rule that they should follow. Shen-Wei confessed in the third interview that he had no idea of what mathematical thinking is really all about. Actually, in Shen-Wei and Win-Pei's minds, development of mathematical knowledge is heavily tied to practical needs, and abstract mathematics will perish after all:

Interviewer: In your point of view, how was mathematical knowledge originated and developed?
Interviewee: It evolved slowly. It was constantly studied along with other daily needs.
Interviewer: Could it be that certain fields of mathematics is invented without daily demand?
Interviewee: It should fit daily demand!
Interviewer: What if some kind of mathematics that has nothing to do with daily living has been created?
Interviewee: That is of no use and will be
forgotten eventually!
(Win-Pei, Third Interview)

With such instrumentalist views in mind, they asserted that mathematical knowledge is invented and may contain flaws.

Table 4
Rating of performance on the standardized problems.

	Fall midterm	Fall final	Spring midterm	Spring final
Chun-Pei	M	H	H	H
Hwa-Chan	H	H	H	H
Shen-Wei	M	L	M	L
Win-Pei	M	L	L	M

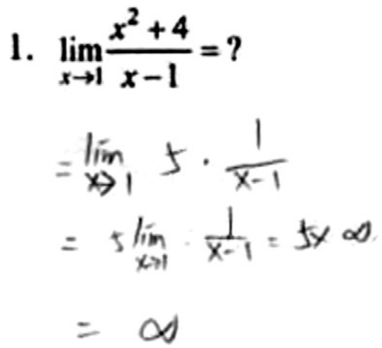


Fig. 2. Shen-Wei's work on finding the limit.

4.2. Performance on the standardized problems

Table 4 shows the four participants' performance on the standardized problems of each examination. It can be seen that, with Fall midterm as an exception, Chun-Pei and Hwa-Chan, both holding relatively sophisticated epistemological beliefs, significantly outperformed Shen-Wei and Win-Pei on these routine exercises. The result suggests Chun-Pei and Hwa-Chan were more likely to be familiar with textbook content and able to make use of learned knowledge at their disposal.

A further investigation of Shen-Wei and Win-Pei's works on exam sheets revealed that they could hardly gain a correct concept image of definitions and demonstrated difficulty in implementing algebraic transformation. Instead, a naïve intuition was involved in guessing the answer and making up the procedure to fit that answer. Fig. 2 shows Shen-Wei's work in finding the limit of $(x^2+4)/(x-1)$ as x approaches 1. His final answer was correct, but was yielded by an inappropriate procedure.

4.3. Performance on the semi-open problems

Table 5 shows unexpected consequences as well as consistent results. Compared to the case of standardized problems, Chun-Pei maintained high or moderate performance on all ten semi-open problems and Shen-Wei still showed a lot of struggle in handling these tasks, requiring not only creative approaches but also a full understanding of calculus content. Results particularly revealed that Shen-Wei did not exhibit high engagement throughout and, therefore, was hardly motivated by his own strategy.

On the other hand, Chun-Pei displayed a greater desire to solve assigned tasks and employed a variety of strategies, though not all worked well. For instance, when asked to prove the area of a circle is πr^2, in addition to using the circle-cutting approach (Fig. 3), he attacked it by exploring the relationship between a square and its inscribed circle (Fig. 4). Though a logical error occurred in his reasoning, the extra effort manifested his attempt to solve the problem.

Results also suggest Hwa-Chan and Win-Pei's cases should receive further investigation. Contrary to his outstanding accomplishment on standardized problems, Hwa-Chan seemed to struggle in resolving these semi-open tasks. Despite his moderate level of engagement, limited use of learned knowledge and a shortage of employing strategies resulted in his low success rate. He exhibited greater motivation in doing area and volume problems, yet showed less will to deal with problems of the infinite series. This situation was somewhat unanticipated for two reasons. First, he had done quite well on typical infinite series problems drawn from the textbook, such as evaluating the infinite sum and determining the convergence of infinite sequence and series. Second, on the Fly Problem, he was on the right rack by calculating the fly's flying time on

Table 5
Rating of performance on the semi-open problems.

	Engagement	Knowledge	Strategy	Completeness
Chun-Pei	7H, 3M, 0L	6H, 4M, 0L	7H, 3M, 0L	6H, 4M, 0L
Hwa-Chan	4H, 5M, 1L	3H, 4M, 3L	1H, 5M, 4L	1H, 6M, 3L
Shen-Wei	0H, 7M, 3L	0H, 4M, 6L	0H, 2M, 8L	0H, 3M, 7L
Win-Pei	5H, 2M, 3L	0H, 5M, 5L	0H, 3M, 7L	0H, 5M, 5L

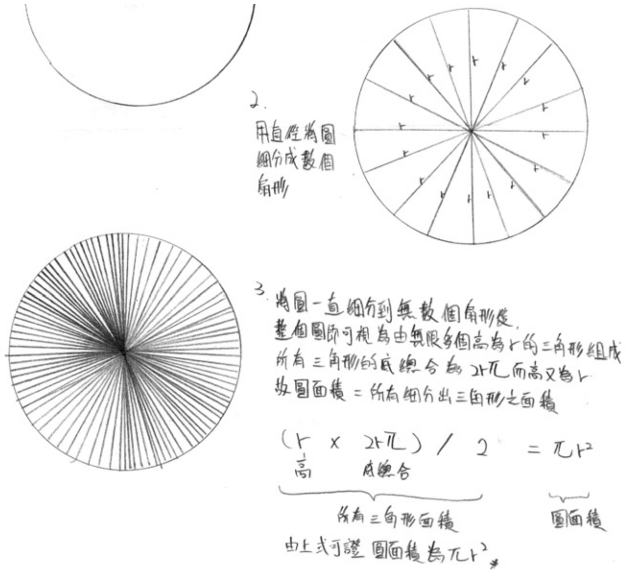


Fig. 3. Chun-Pei's first proof of the area of a circle is πr^2.

the basis of how long the two approaching trains would bump into each other, as opposed to falling into a potential trap of evaluating complicated infinite sums. He appeared to be more in favor of problems directly solved by learned strategies, whereas he showed deficiency on those requiring proof or explanation.

Another unexpected case was Win-Pei's active engagement on semi-open problems. Her achievement on the standardized tests was below average, yet she demonstrated enthusiasm in trying possible approaches to get these semi-open problems completed. Not being able to sophisticatedly utilize learned knowledge and create her own strategies, she failed to completely resolve any of the problems. Taking the Salesman Problem as an example, it was hoped that students may connect the problem with The Intermediate Value Theorem, but Win-Pei made an assumption not given and employed naïve intuition to answer the problem, treating the whole problem out of mathematical context. Hwa-Chan and Win-Pei's frustrated outcomes were caused by their insufficiency in "knowledge" and "strategy." Note that they were allowed 1–2 weeks to accomplish the tasks and seeking outside sources was permitted. It appeared their initial willingness of engagement was not followed by active pursuit of extracurricular information and planning.

4.4. Behavior on the pattern-finding problems

The researcher also probed participants' problem solving behavior through observing how promptly they reacted to interview tasks and developed their own strategies. Three pattern-finding problems (Tower of Hanoi-1, Tower of Hanoi-2,

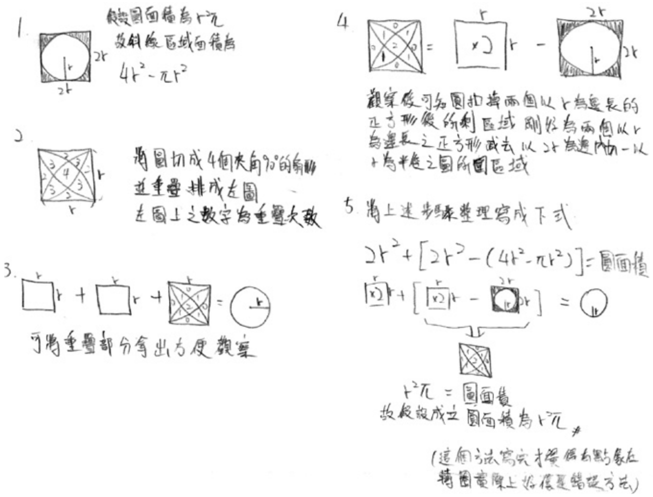


Fig. 4. Chun-Pei's second proof of the area of a circle is πr^2.

Table 6
Hwa-Chan's problem solving behavior on the Tower of Hanoi-2.

Time	Description of Hwa-Chan's behavior and conversation
2′50″	Finished 6-disc case.
3′49″	Got stuck and shifted randomly.
4′16″	Q: Did you count how many movements you made?
	A: Yes, I did. It's about 70 or more, maybe 80.
5′16″	Finished 7-disc case.
7′07″	Suddenly found a mistake and tried to get back.
8′34″	Got lost again.
9′18″	Seemingly back on the right track.
10′12″	Finished 7-disc case and began to work on the 8-disc case, but the second large disc was isolated. . .
12′08″	Finished!
	Q: Did you count the number of moves?
	A: . . .Around 128.
	Q: How did you get this number?
	A: Because one move for one disc, three moves for two discs, seven moves for three discs, fifteen moves for four discs. . .(counting and demonstrating).
	Q: Then why 128?
	A: I got 1,3,7,15,. . .there must be a relationship among them. . .
	Q: Do you want to write them down?
	A: . . .add them all. . .(wrote down all numbers but did not recognize the patterns).

and Three-order Magic Square) require only slight mathematical background but need meta-cognitive awareness to keep their thinking on track. Not too surprisingly, Chun-Pei finished all three interview tasks without too much struggle. However, the other three participants' performance disclosed several phenomena deserving more attention.

Hwa-Chan's case: Hwa-Chan confessed he had done the Magic Square Problem before and quickly filled in all numbers correctly. On the problem of Tower of Hanoi, however, he fell into a swamp, although he claimed that he had also tried this before. In the first interview, Hwa-Chan completed perfectly the shifting of the 6 discs to another pole in 2 min and 28 s, yet got stuck later. At the time of 10′58″, he tried to go back to the 5-disc case and start over, but still was not successful. At this moment, he began to show anxious and impatient behavior, the researcher suggested he restart from the 6-disc case. At 15′07″, he made it to the 7-disc case but soon encountered other difficulties once again. From then on, Hwa-Chan's action turned random and thus he was unable to finish the task as a result.

Four months later, Hwa-Chan was asked to do the problem of Tower of Hanoi-2. In addition to shifting all 8 discs to another pole, he was required to minimize and count the movements. Similar to the previous time, he proficiently shifted 5 discs in 1 min, yet soon got stuck when working on the 6-disc case. He struggled a lot and shifted discs in random order without following any rule. It took him another 1′51″ to get the 6-disc case completed but he soon was stuck again. Table 6 shows annotations of Hwa-Chan's behavior and the conversation between him and the interviewer.

Though Hwa-Chan completed shifting all discs, he apparently did not consistently keep tracing all the moves and finally got lost in identifying the pattern. On the problems of Tower of Hanoi-1 and 2, Hwa-Chan initially worked very well by relying on his memory, as he said he had done this before. On the problem of Tower of Hanoi-1, he spent 15 min and 7 s on the 7-disc case and only 5 min and 16 s was required on Tower of Hanoi-2. It appears he made a lot of progress on the basis of previous experience. Whereas he turned out to be on a wild-goose-chase and went around in circles lasting 5 min (from 5′16″ to 10′12″). Furthermore, he seemingly was not tracking the exact number of moves, which is the main task of Tower of Hanoi-2. On the basis of the analysis of the videotape, it can be seen that Hwa-Chan's failure on Tower of Hanoi-1 and predicament in Tower of Hanoi-2, to a great extent, was due to his unsuccessfully controlled behavior and being unaware of the operating pattern.

Win-Pei and Shen-Wei's cases: Win-Pei and Shen-Wei demonstrated similarity in many aspects. Not only did they both show naïve epistemological beliefs about mathematics but also displayed similar behavior in the interview tasks. They both successfully accomplished Tower of Hanoi-1 and 2 but failed to complete the Magic Square Problem.

Among all participants, Win-Pei spent the longest time (24 min and 52 s) in shifting 8 discs. In the first interview, she carefully made each move and stopped from time to time to get out of sticky situations. Asked to identify the minimum count for shifting all discs in the second interview, she advertently wrote down each count on the working sheet when finished moving each disk and gazed at the tower and disks for a while. At the time of 6′43″, Win-Pei had finished moving 5 discs and had written down several "key numbers" so the interviewer decided to intervene at this moment:

Q: What do these numbers mean (the interviewer pointed to the numbers of 3, 7, 15, and 31 on the working sheet)?
A: The count for each case. Two discs need 3 moves; 3 discs need 7 moves; 4 discs 15 moves; and 5 discs 31 moves.
Q: Then what do the numbers 4, 8, 16, and 32 mean? (She had also noted 4, 8, and 16 beside 3, 7, 15, and 31.)
A: The difference between 3 and 7 is 4; between 7 and 15 is 8; between 15 and 31 is 16.
Q: So you guessed the next difference would be 32? (She marked 63 on the working sheet earlier for the 6-disc case.)
A: Yes!
Q: What about 7 discs?

A: . . .(pondering) It needs 127 moves.
Q: What about 8 discs?
A: It is 255.
Q: Are you confident with your conjecture?
A: . . .(pondering) I might be wrong if I moved it wrong in the beginning.
Q: How much confidence do you have in your answer?
A: I think I was doing it right.
Q: Yes, you were doing it right, but how did you know it could be done in this way?
A: Just started from the simplest case.
Q: So you knew to find the rule in the very beginning?
A: Yes!

Unlike Hwa-Chan, rushing to complete the 8-disc case without tracing each count, Win-Pei, who had never done this before, appeared to be aware of a pattern out there behind the problem and therefore kept tracking each count. Such a "start from the basic" and "look for a pattern" problem solving behavior was also seen in Shen-Wei's case. He noted each count he had until the 6-disc case, and then hypothesized that 255 moves in total are required for the 8-disc case. By figuring out that each count is one more than twice the previous count, he developed another pattern different than that of Win-Pei (Table 7). Contrary to their humdrum performance on the standardized and semi-open problems, both Win-Pei and Shen-Wei unexpectedly exhibited expert-like behavior on the task of Tower of Hanoi. The situation, however, was the opposite in the third interview.

Similar to the Tower of Hanoi, Win-Pei had a good start with the problem 3 × 3 Magic Square. She placed the number 5 in the center and then assigned all the other numbers to form four pairs: (1, 9), (2, 8), (3, 7), and (4, 6). These signs seemingly suggested she was on the right track, but subsequent actions made the whole thing go awry. Win-Pei wrote down the number 13 and attempted to pair the numbers 1 to 9 (except 5) off to make 13. She was stuck for a long while and the interviewer intervened to ask her about the reason for placing 5 in the center and what the number 13 represented. She said because 5 is the medium of the numbers 1 through 9 and she just felt that the sum of each row and column might be 13, but was unable to give any meaningful reason for the choice of the number 13. After conversation, Win-Pei continued to unconsciously pair numbers off to make 13 and set the four pairs, (1, 9), (2, 8), (3, 7), and (4, 6), aside. Her perfect beginning turned out to be a lucky but insensible accident.

Shen-Wei confessed he had worked on the Magic Square before, but was unable to recall the rule. At the time of 1′40″, he suddenly got an insight followed by placing 5 in the center and filling all the other numbers into the grid. He was convinced he had solved the problem, but that was not the case:

Q: Are you done?
A: Yes!
Q: Does the sum of each row and column equal?
A: Do they need to be the same? (He then erased all the other numbers except number 5.)
Q: Why did you keep 5 there?
A: Five should be in the center, because the sum of each row and column is 15. (He appeared to recall the rule yet showed no proof to the cause.)
Q: Why?
A: I've heard that before. . .I think it is 15. . .I am kind of forgetting.
Q: Why 15?
A: I have no idea. . .I don't remember.

Table 7
Win-Pei and Shen-Wei's pattern for Tower of Hanoi.

No. of Discs	1	2	3	4	5	6	7	8
Win-Pei's	**1**	**3**	**7**	**15**	**31**	**63**	**127**	**255**
Pattern		=**1**+2	=**3**+4	=**7**+8	=**15**+16	=**31**+32	=**63**+64	=**127**+128
Shen-Wei's	**1**	**3**	**7**	**15**	**31**	**63**	**127**	**255**
Pattern		=**1**×2+1	=**3**×2+1	=**7**×2+1	=**15**×2+1	=**31**×2+1	=**63**×2+1	=**127**×2+1

Q: Then why is 5 placed in the center?
A: I just remember so.
Q: But you don't know why?
A: Yes!

Five minutes later, Shen-Wei was still struggling and the interviewer reminded him the rule he just mentioned. Nonetheless, he still fell into confusion and was not aware that what he needed was to pair the numbers 1–9 (except 5) to make 10. His incomplete memory and the pressure of recalling the rules apparently restricted his thinking.

5. Discussion

Owing to a widely accepted working definition of beliefs and the theoretical foundation for the study of beliefs are lacking, it has often been questioned if beliefs are believable (Lester, 2002; Mason, 2004). Instead of exploring general belief systems, the present study restricted the main focus to college students' epistemological beliefs about mathematics, and investigated how they related to students' mathematical performance and behavior in different contexts. Results yielded from the survey of students' performance on standardized tests, semi-open problems, and their behaviors on pattern-finding tasks, suggest mixed consequences.

Through the whole academic year, students espousing sophisticated and active beliefs about mathematical knowledge and thinking (Chun-Pei and Hwa-Chan) had done quite well on all standardized tests; those who held naïve and conventional epistemological beliefs in mathematics (Win-Pei and Shen-Wei) had shown poor performance. The situation, however, turned out to be somewhat complicated as the investigation advanced. Unlike Chun-Pei, who consistently behaved enthusiastically and mostly succeeded on all kinds of problems, Hwa-Chan displayed only moderate engagement in semi-open problems and lost his control on two Tower of Hanoi tasks. He rushed into the task without bearing any patterns in mind and failed to finish as a result. His personal epistemological beliefs in mathematics seemingly played a minor role in these cases. In contrast, Win-Pei and Shen-Wei demonstrated expert-like behavior on the Tower of Hanoi in which they carefully made each move and noted the pattern to reach the end. Limited views of the nature of mathematics did not hamper their performance on this pattern-finding task. Such a seemingly conflicting result raises several critical issues needing reconsideration. What do we mean by a good problem solver in mathematics? What makes a problem solver perform well in one mathematical context but poorly in another? What accounts for the success or failure of the problem solving attempt? What is more important in terms of the purpose of the present study is the question, are beliefs really believable?

In a broad sense, Hwa-Chan, Win-Pei, and Shen-Wei can be categorized as good problem solvers in certain aspects, though Hwa-Chan showed wild-goose-chase behavior and Win-Pei and Shen-Wei were less capable of dealing with content-laden tasks. Win-Pei's and Shen-Wei's success on the Tower of Hanoi was attributed to their persistence and moderate understanding of heuristic strategies: starting from the basic and looking for a pattern. They appeared to be unable to transfer this capability to standardized and semi-open problems, which requires a more solid content knowledge. Win-Pei's case is particularly noteworthy. Despite her failure, she had shown a willingness to try semi-open tasks, but was hampered by the shortage of knowledge and strategies. It seems Win-Pei's mathematical performance on test items and homework problems was heavily compressed by content knowledge, and her inconsistent behavior in interview tasks was connected to her naïve make-sense epistemology in which the approval of truth depends upon intuitive sense—"the intuitive feel of fit suffices" (Perkins, Allen, & Hafner, 1983, p. 186).

Hwa-Chan's behavior, nonetheless, suggests another way of thinking. In Schoenfeld's (1985) research-based framework for interpreting problem solving behavior, there is an implicit link between belief systems and control behavior. Recent studies also indicate that, while working on the non-routine problem, students' behavior and control decisions were more related to their personal belief system as opposed to well-defined problem solving (Bendixen & Hartley, 2003; Lerch, 2004). All these claims fail to explain Hwa-Chan's case. His sophisticated epistemological beliefs and out-performance on standardized test items reveal that he was by no means a naïve thinker. However, his meta-cognitive failure on the problem of Tower of Hanoi makes no link to his sophisticated personal epistemology. Namely, his personal active and sophisticated epistemological beliefs in mathematics were less likely to play a critical role in these cases. It seems the connection between belief systems and control behavior is not only implicit but indirect. Problem solvers tend to re-conceptualize the problem in the midst of working and such a process of re-conceptualization is an internal dialogue regarding the way that their solutions evolve (Schoenfeld, 1985). Competent problem solvers should constantly demonstrate good control behavior through arguing with themselves as they work to avoid meta-cognitive failure. Hwa-Chan seemingly fell into '*meta-cognitive blindness*' (Goos, 2002) by persistently using inadequate strategies, preventing him from recognizing any pattern. Win-Pei's failure on the Magic Square could be attributed to the occurrence of '*meta-cognitive mirage*' (Goos, 2002), misleading her to abandon a useful and correct strategy.

5.1. *Limitations and further study*

Mason (2004) questioned the believability of beliefs and proposed that the habits formed through prior experiences, the natural response to problem situations, and the result of rational choices, are three potential variables accounting for peoples' actions. Nonetheless, these three constructs are by no means irrelevant to the structure of a belief system. On the basis of observational data, it appears that beliefs played a more reliable role within the well-structured context but lost credibility in non-conventional tasks. The complicated phenomena may be caused by a distinction between beliefs that are in fact true and those that are felt to be true, which was not researched in this study and hence is a limitation of this report. Besides, difficulties in clearly identifying an individual's beliefs and behaviors within a specific context have been well recognized, and belief systems might show consistency within a specific context but not in another (Hofer, 2001). Though several instruments had been employed to explicate participating college students' epistemology in mathematics to a feasible extent, the innate restriction of questionnaires and interviews might fall short of probing the complicated nature of beliefs.

The present 1-year study investigating participants' epistemological beliefs of mathematics across three different time slots should be treated as relatively reliable. Even so, their personal epistemology gave only partial account to their mathematical performance and behavior. Are beliefs really believable? Or might it be that the inconsistencies were caused by the instruments employed in the present study? I believe the belief puzzle could be solved only when the aforementioned ambiguity can be clarified.

Acknowledgements

The author wishes to thank anonymous reviewers for their insightful comments. This paper is part of a research project funded by the National Science Council of Taiwan (NSC 93-2521-S-167-001). The views and opinions expressed in this paper are those of the author and not necessarily those of NSC.

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