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Age replacement policy with lead-time for a system subject to non-homogeneous pure birth shocks

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ABSTRACT

A system is subject to shocks that arrive according to a non-homogeneous pure birth process. Whenever a shock occurs, the system enters one of the two types of failure states. Type I failure (minor failure) is fixed by a minimal repair. Type II failure (catastrophic failure) is removed by a replacement. We consider an age replacement policy which replaces the system whenever its age reaches T and a spare for replacement is available. The optimal cost minimization age T^* is derived under a cost structure. We demonstrate that this model includes more realistic factors and is a generalization of several previous models in the literature.

1. Introduction

A system is subject to shocks. Whenever a shock occurs, the system enters one of the two types of minor and major failure states, which require minimal repair and replacement actions, respectively. These failures result in interruptions of the system operation and repair/replacement costs. Therefore, it is important to study various maintenance policies to minimize the operating cost and the risk of a catastrophic breakdown. The maintenance policy with both replacements and minimal repairs was first introduced by Barlow and Hunter [1]. In the past several decades, maintenance and replacement problems were extensively studied in the literature. A well-known maintenance policy is the classical age-replacement policy, in which an operating system is replaced at the time of failure or at age T, whichever occurs first. The extension of such a policy has been considered by many researchers including Bai and Yun [2], Berg et al. [3], Block et al. [4,5], Chen and Savits [6], Chien and Sheu [7], and Chien et al. [8], Sheu [9], Sheu and Chien [10], Sheu and Griffith [11], Sheu et al. [12,13], Sheu and Kuo [14]. Wang [15] summarized, classified, and compared a variety of maintenance policies for deteriorating systems.

Most of the past studies modeled the shocks as a non-homogeneous Poisson process (NHPP). Note that the NHPP shocks only depend on the system's age, but not on the number of failures. However, many practical systems deteriorate with age as

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well as the number of repairable failures. To incorporate this feature, we model the shocks as a non-homogeneous pure birth process (NHPBP), which is defined as follows.

Definition 1. If a counting process $\{N(t); t \geqslant 0\}$ is a non-homogeneous continuous time Markov process with following conditions:

(i) $N(0) = 0$,
(ii) $P\{N(t+h) - N(t) = 1 | N(t) = k\} = r_k(t)h + o(h)$,
(iii) $P\{N(t+h) - N(t) \geqslant 2 | N(t) = k\} = o(h)$,
(iv) the process has independent increments,

then the process is called a non-homogeneous pure birth process (denoted by NHPBP) with the intensity function $\{r_k(t), k = 0, 1, 2, \ldots\}$ and mean value function $\Lambda_k(t) = \int_0^t r_k(u)du$. □

From the above definition, it is obviously that the NHPBP has a failure rate which depends on both the system's age and the number of shocks. Thus, the maintenance model with the NHPBP shocks is a generalization of the existing models and can be applied in production, insurance, epidemiology, and load-sharing systems.

The rest of this paper is organized as follows. Section 2 describes the system we study and makes the assumptions. Section 3 formulates the maintenance model and develops the long-term average cost function. Section 4 presents the cost minimization age replacement policy. Section 5 discusses the computation of the optimal policy and provides an algorithm. Finally, Section 6 shows that many classical models are the special cases of our model.

2. The system and assumptions

In this section, we describe the system and present the assumptions.

2.1. Operation of the system

Consider a system subject to NHPBP shocks, each shock causes the system to fail in one of the two types. A type I failure (minor) is fixed by a minimal repair and a type II failure (major) is cleared by an emergent replacement. Whenever the operating system's age reaches T, it is replaced with an available new system. T is the decision variable of this study.

The probability of each failure type is assumed to depend on the number of type I shocks since the last replacement, denoted by M. Let $\overline{P}_k = P(M > k)$ is the probability that the first k shocks of the system are all type I. The domain of $\overline{P}_k$ is $\{0, 1, 2, \ldots\}$ and $1 = \overline{P}_0 \geqslant \overline{P}_1 \geqslant \overline{P}_2 \geqslant \ldots$. Denote by $\{\overline{P}_k\}$ the sequence of probabilities $\overline{P}_k$'s and let $p_k = P(M = k) = \overline{P}_{k-1} - \overline{P}_k = \overline{P}_{k-1}(1 - \overline{P}_k/\overline{P}_{k-1})$ with domain $\{1,2,3,\ldots\}$. Therefore, the conditional probability that the kth shock results in a type I failure (or a type II failure) is $q_k = \overline{P}_k/\overline{P}_{k-1}$ (or $\theta_k = 1 - q_k = 1 - \overline{P}_k/\overline{P}_{k-1}$). Furthermore, we assume that the random variable M is independent of the shock process $\{N(t); t \geqslant 0\}$.

It is assumed that a new system (spare unit) for replacement is delivered upon order, and the lead-time between ordering and receiving the new system is a random variable. The random lead time L has cumulative distribution function (c.d.f.) $G(t)$, probability density function (p.d.f.) $g(t)$, survival function (s.f.) $\overline{G}(t)$, and finite mean $E(L) = \mu$. This assumption is a unique feature of our model and is particular important in the situation where the system is very expensive. This is because, except for the first order placed at time 0, ordering it only upon failure can significantly reduce the cost of capital.

2.2. Maintenance of the system

The detailed maintenance scheme for the system can be described explicitly as follows. A new system is put in operation and a new order has been placed at time 0. If the ordered system is received before the occurrence of a type II failure or age T, the replacement can be made immediately whenever needed. Otherwise, the type II failure or age T replacement cannot be made until the ordered system is received.

There are four mutually exclusive and collectively exhaustive cases for the system replacement with different costs.

- *Case 1.* If the ordered system arrives and no type II failure occurs before time T, then the operating system is replaced at age T with a cost c_1 (preventive replacement).
- *Case 2.* If the ordered system arrives after time T and no type II failure occurs before the arrival of the ordered system, then the operating system is replaced at the order arrival instant with a cost c_2 (delayed preventive replacement).
- *Case 3.* If the ordered system arrives before a type II failure which occurs before time T, then the failed system is replaced with the new system at the type II failure instant at a cost c_3 (corrective replacement).
- *Case 4.* If the ordered system arrives after a type II failure which occurs before T, then the failed system is replaced at the order arrival instant with a cost c_4 (delayed corrective replacement).

2.3. Cost structure and additional assumptions

The cost of the ith repair at age t is a non-decreasing function of age and the number of repairs, denoted by $\phi(C(t), c_i(t))$, where $C(t)$ is the age-dependent random part, $c_i(t)$ is the deterministic part. Hence, the expected repair cost is $\alpha_i(t) = E_{C(t)}[\phi(C(t), c_i(t))]$. Denote the c.d.f., p.d.f. and mean of $C(t)$ by $W_t(x)$, $w_t(x)$ and $E[C(t)]$, respectively.

The replacement cost order $c_1 = c_2 < c_3 \leqslant c_4$ is assumed, where $c_3 > c_2$ means that the corrective replacement cost is greater than the preventive replacement cost; and $c_4 \geqslant c_3$ means that c_4 is higher due to the operation interruption affecting the customer service negatively.

Furthermore, let c_h be the cost per unit time for holding a new idle system (i.e., the spare system), and let c_s be the cost per unit system downtime.

Finally, we also make the following assumptions:

(1) The system is monitored continuously and all failures are detected immediately.
(2) Repairs and replacements are completed instantaneously.
(3) After a replacement, the system becomes new and the process starts again.

The assumption regarding the random lead time for delivering the spare unit has been made in many past studies including Chien [16,17], Chien and Chen [18,19], Kaio and Osaki [20], Nakagawa and Osaki [21], Osaki [22], Osaki and Yamada [23], Osaki et al. [24], Sheu and Liou [25], Thomas and Osaki [26,27].

3. Model formulation and analysis

Define the following transition probability at time t given $N(0) = 0$:

$$P_k(t) = P\{N(t) = k | N(0) = 0\}, k = 0, 1, 2, \ldots, \tag{1}$$

which can be further computed as

$$P_k(t) = e^{-\Lambda_k(t)} \int_0^t e^{\Lambda_k(x_{k-1}) - \Lambda_{k-1}(x_{k-1})} r_{k-1}(x_{k-1}) \int_0^{x_{k-1}} e^{\Lambda_{k-1}(x_{k-2}) - \Lambda_{k-2}(x_{k-2})} r_{k-2}(x_{k-2}) \cdots \int_0^{x_2} e^{\Lambda_2(x_1) - \Lambda_1(x_1)} r_1(x_1) \times \int_0^{x_1} e^{\Lambda_1(x_0) - \Lambda_0(x_0)} r_0(x_0) \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{k-2} \mathrm{d}x_{k-1}, \tag{2}$$

for $k = 1, 2, \ldots$ and $P_0(t) = e^{-\Lambda_0(t)}$ (see Sheu et al. [28] for the detailed derivation). Also note that

$$\frac{\mathrm{d}}{\mathrm{d}t} P_k(t) = -r_k(t) P_k(t) + r_{k-1}(t) P_{k-1}. \tag{3}$$

If we assume that there is no planned replacement and lead time is zero (i.e., $T = \infty$ and $L = 0$) and let Y be the time interval between two successive type II failure replacements, the survival function of Y can be written as

$$\overline{H}(t) = \sum_{k=0}^{\infty} P(N(t) = k, M > k) = \sum_{k=0}^{\infty} P_k(t) \overline{P}_k. \tag{4}$$

It follows from (4) that the density function $h(t) = -\mathrm{d}\overline{H}(t)/\mathrm{d}t$ is

$$h(t) = \sum_{k=0}^{\infty} P_k(t) r_k(t) p_{k+1}. \tag{5}$$

Thus, the type II failure rate at time t is $r_H(t) = h(t)/\overline{H}(t)$.

Let U_j denote the length of successive replacement cycle j ($j = 1, 2, 3, \ldots$), and V_j the operational cost during U_j. Thus $\{(U_j, V_j)\}$ constitutes a renewal reward process. If $D(t)$ is the expected cost of operating the system over the time interval $[0, t]$, then it is well known that

$$\lim_{t \to \infty} \frac{D(t)}{t} = \frac{E(V_1)}{E(U_1)}, \tag{6}$$

(see, e.g., Ross [29]; p. 52)). We shall denote the right-hand side of (6) by $B(T)$. For the infinite horizon case, we want to find optimal T^* which minimize $B(T)$, the total expected long-run cost per unit time.

The following lemma is needed in deriving $B(T)$.

Lemma 1. *If a type II failure does not occur in* $[0, t]$*, let* $\Phi(t)$ *be the total minimal repair cost incurred over* $[0, t]$*, then*

$$E[\Phi(t)] = \int_0^t \sum_{i=1}^{\infty} \alpha_i(y) P_{i-1}(y) r_{i-1}(y) \overline{P}_i \frac{1}{\overline{H}(y)} dy, \tag{7}$$

where $\alpha_i(y) = E_{C(y)}[\phi(C(y), c_i(y))]$ *is finite for all* $y \geqslant 0$ *and* $i \geqslant 1$.

(See Appendix A for the proof).

Let $Y_1, Y_2, \ldots$ be the independent copies of Y (independent and identically distributed random variable sequence). According to the replacement scheme described in Section 2, we have

$$U_1 = \begin{cases} T, & \text{if } L \leqslant T < Y_1, \\ L, & \text{if } T \leqslant L < Y_1, \\ Y_1, & \text{if } L \leqslant Y_1 \leqslant T, \\ L, & \text{if } Y_1 \leqslant L, \end{cases} \tag{8}$$

and

$$V_1 = \begin{cases} c_1 + \Phi(T) + c_h(T - L), & \text{if } L \leqslant T < Y_1, \\ c_2 + \Phi(L), & \text{if } T \leqslant L < Y_1, \\ c_3 + \Phi(Y_1) + c_h(Y_1 - L), & \text{if } L \leqslant Y_1 \leqslant T, \\ c_4 + \Phi(Y_1) + c_s(L - Y_1), & \text{if } Y_1 \leqslant L. \end{cases} \tag{9}$$

By (8), the expected length of a replacement cycle is given by

$$\begin{aligned} E(U_1) &= T \cdot \overline{H}(T) \cdot G(T) + \textstyle\int_T^\infty t \cdot \overline{H}(t) \mathrm{d}G(t) + \int_0^T t \cdot G(t) \mathrm{d}H(t) + \int_0^\infty t \cdot H(t) \mathrm{d}G(t) \\ &= T \cdot \overline{H}(T) \cdot G(T) + \textstyle\int_T^\infty t \mathrm{d}G(t) + \{[t \cdot G(t) \cdot H(t)|_0^T] - \int_0^T H(t) \mathrm{d}[tG(t)]\} + \int_0^T t \cdot H(t) \mathrm{d}G(t) \\ &= T \cdot \overline{H}(T) \cdot G(T) - \textstyle\int_T^\infty t \mathrm{d}\overline{G}(t) + T \cdot G(T) \cdot H(T) - \int_0^T H(t) \cdot G(t) \mathrm{d}t - \int_0^T t \cdot H(t) \mathrm{d}G(t) + \int_0^T t \cdot H(t) \mathrm{d}G(t) \\ &= T \cdot G(T) - \{[t \cdot \overline{G}(t)|_T^\infty] - \textstyle\int_T^\infty \overline{G}(t) \mathrm{d}t\} - \int_0^T [1 - \overline{H}(t)] G(t) \mathrm{d}t \\ &= T \cdot G(T) + T \cdot \overline{G}(T) + \textstyle\int_T^\infty \overline{G}(t) \mathrm{d}t - \int_0^T G(t) \mathrm{d}t + \int_0^T \overline{H}(t) \cdot G(t) \mathrm{d}t \\ &= T + \textstyle\int_T^\infty \overline{G}(t) \mathrm{d}t - T + \int_0^T \overline{G}(t) \mathrm{d}t + \int_0^T \overline{H}(t) \cdot G(t) \mathrm{d}t \\ &= \textstyle\int_0^\infty \overline{G}(t) \mathrm{d}t + \int_0^T \overline{H}(t) \cdot G(t) \mathrm{d}t \\ &= \textstyle\int_0^T \overline{H}(t) \cdot G(t) \mathrm{d}t + \mu. \end{aligned} \tag{10}$$

Similarly, by using the integration by parts and the double integral as in the calculation of $E(U_1)$, the total expected cost in a replacement cycle can be expressed as

$$\begin{aligned} E(V_1) = {} & c_1 \cdot \overline{H}(T) \cdot G(T) + c_2 \cdot \int_T^\infty \overline{H}(t) \mathrm{d}G(t) + c_3 \cdot \int_0^T G(t) \mathrm{d}H(t) + c_4 \cdot \int_0^\infty H(t) \mathrm{d}G(t) \\ & + \int_0^T \sum_{i=1}^\infty \alpha_i(t) P_{i-1}(t) r_{i-1}(t) \overline{P}_i G(t) \mathrm{d}t + \int_0^\infty \int_0^t \sum_{i=1}^\infty \alpha_i(y) P_{i-1}(y) r_{i-1}(y) \overline{P}_i \mathrm{d}y \mathrm{d}G(t) + c_h \cdot \int_0^T \overline{H}(t) \cdot G(t) \mathrm{d}t \\ & + c_s \cdot \int_0^\infty \overline{G}(t) \cdot H(t) \mathrm{d}t. \end{aligned} \tag{11}$$

The detailed derivation of (11) is presented in Appendix B. Introducing the notation of $\zeta(T) = E(V_1)$ to indicate that $E(V_1)$ is a function of T, it follows from (6) that

$$B(T) = \frac{\zeta(T)}{\int_0^T \overline{H}(t) \cdot G(t) \mathrm{d}t + \mu}. \tag{12}$$

4. Optimal policy

To determine the optimal age T^* that minimizes the expected cost rate function $B(T)$, we can taking the first-order derivative of $B(T)$ with respect to T and setting it to be zero. Thus, by the expected cost rate function that given in (12), $\mathrm{d}B(T)/\mathrm{d}T = 0$ if and only if

$$\left[(c_3-c_2)r_H(T)+\left(\sum_{i=1}^{\infty}\alpha_i(T)P_{i-1}(T)r_{i-1}(T)\overline{P}_i\right)\frac{1}{\overline{H}(T)}+c_h\right]\left[\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t+\mu\right]$$
$$=c_1\cdot\overline{H}(T)G(T)+c_2\cdot\int_T^{\infty}\overline{H}(t)\mathrm{d}G(t)+c_3\cdot\int_0^T G(t)\mathrm{d}H(t)+c_4\cdot\int_0^{\infty}H(t)\mathrm{d}G(t)+\int_0^T\sum_{i=1}^{\infty}\alpha_i(t)P_{i-1}(t)r_{i-1}(t)\overline{P}_iG(t)\mathrm{d}t$$
$$+\int_0^{\infty}\int_0^t\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\overline{P}_i\mathrm{d}y\mathrm{d}G(t)+c_h\cdot\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t+c_s\cdot\int_0^{\infty}\overline{G}(t)\cdot H(t)\mathrm{d}t, \tag{13}$$

or

$$\left[(c_3-c_2)r_H(T)+\left(\sum_{i=1}^{\infty}\alpha_i(T)P_{i-1}(T)r_{i-1}(T)\overline{P}_i\right)\frac{1}{\overline{H}(T)}+c_h\right]\left[\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t+\mu\right]$$
$$=(c_3-c_2)\cdot\int_0^T G(t)\mathrm{d}H(t)+\int_0^T\sum_{i=1}^{\infty}\alpha_i(t)P_{i-1}(t)r_{i-1}(t)\cdot\overline{P}_i\cdot G(t)\mathrm{d}t+c_h\cdot\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t+c_2\cdot\int_0^{\infty}G(t)\mathrm{d}H(t)$$
$$+c_4\cdot\int_0^{\infty}H(t)\mathrm{d}G(t)+\int_0^{\infty}\int_0^t\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\overline{P}_i\mathrm{d}y\mathrm{d}G(t)+c_s\cdot\int_0^{\infty}\overline{G}(t)\cdot H(t)\mathrm{d}t. \tag{14}$$

Let

$$\varepsilon(y)=(c_3-c_2)\cdot r_H(y)+\left(\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\overline{P}_i\right)\frac{1}{\overline{H}(y)}+c_h. \tag{15}$$

Thus, we have the following properties concerned the optimal age replacement policy.

Theorem 1. *Suppose that the NHPBP has the intensity rate function $\{r_i(t), i=0,1,2,\ldots\}$; $r_i(y)$ and $\alpha_i(y)$ are continuous in y; $c_1=c_2<c_3\leqslant c_4$; and $\varepsilon(y)$ is continuous and strictly increasing in y.*

Let $\delta=\zeta_1/\mu$, and $\vartheta=\zeta_2/\left(\int_0^{\infty}\overline{H}(t)\cdot G(t)\mathrm{d}t+\mu\right)$, where ζ_1 and ζ_2 are respectively given by

$$\zeta_1\equiv c_2\cdot\int_0^{\infty}G(t)\mathrm{d}H(t)+c_4\cdot\int_0^{\infty}H(t)\mathrm{d}G(t)+\int_0^{\infty}\int_0^t\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\overline{P}_i\mathrm{d}y\mathrm{d}G(t)+c_s\cdot\int_0^{\infty}\overline{G}(t)\cdot H(t)\mathrm{d}t,$$
$$\zeta_2\equiv c_3\cdot\int_0^{\infty}G(t)\mathrm{d}H(t)+c_4\cdot\int_0^{\infty}H(t)\mathrm{d}G(t)+\int_0^{\infty}\sum_{i=1}^{\infty}\alpha_i(t)P_{i-1}(t)r_{i-1}(t)\overline{P}_iG(t)\mathrm{d}t$$
$$+\int_0^{\infty}\int_0^t\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\overline{P}_i\mathrm{d}y\mathrm{d}G(t)+c_h\cdot\int_0^{\infty}\overline{H}(t)\cdot G(t)\mathrm{d}t+c_s\cdot\int_0^{\infty}\overline{G}(t)\cdot H(t)\mathrm{d}t.$$

Then

(i) If $\varepsilon(0)<\delta$ and $\varepsilon(\infty)>\vartheta$. Then there exists a finite and unique T^* which minimizes $B(T)$; and

$$B(T^*)=\varepsilon(T^*)=(c_3-c_2)\cdot r_H(T^*)+\left(\sum_{i=1}^{\infty}\alpha_i(T^*)P_{i-1}(T^*)r_{i-1}(T^*)\overline{P}_i\right)\frac{1}{\overline{H}(T^*)}+c_h \tag{16}$$

(ii) If $\varepsilon(\infty)\leqslant\vartheta$. Then the optimum replacement policy is $T^*=\infty$: no planned replacement.
(iii) If $\varepsilon(0)\geqslant\delta$. Then the optimum replacement policy is $T^*=0$: replacement is made just after the arrival of the ordered spare.

(See Appendix C for the proof).

Note that in Theorem 1, $T^*=\infty$ means that the optimal replacement policy is no planned replacement and $T^*=0$ means that the optimal replacement policy is to place replacement just after the arrival of the ordered spare.

5. Calculations

The terms $(c_3-c_2)\cdot r_H(y)$ and $\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\overline{P}_i/\overline{H}(y)$ in (13), (14) and (15) can be regarded as the s-expected marginal costs of replacement and repair, respectively. And $\varepsilon(y)$ can be regarded as the s-expected marginal cost of the age replacement policy at age y. As in Berg and Clèroux [30], the s-expected marginal cost of this policy at age y is expressed as a linear combination of its component costs. The optimal T^*, which minimizes $B(T)$, has to satisfy the condition $B(T)=\varepsilon(T)$, a well-known principle in economics. In practice, we can determine T^* as follows: draw the functions $B(T)$ and $\varepsilon(T)$ on the same graph and find the intersection point of the two functions. If $\varepsilon(y)$ is continuous and strictly increasing, then there exists a unique intersection point.

Here we present the algorithm that can be used to compute the optimal T^* and $B(T^*)$, numerically.

Algorithm.

Input: $c_1, c_2, c_3, c_4, c_h, c_s, \{\overline{P}_k\}, r_k(\cdot), \Lambda_k(\cdot), \alpha_k(\cdot), G(\cdot)$
Step 1. Compute $p_k, P_k(t), \overline{H}(t)$ and $h(t)$ are defined by (2), (4), and (5), respectively.
Step 2. Find the unique solution T^* that minimizes $B(T)$ (i.e., the solution satisfies (14)).
Step 3. Compute $B(T^*)$ as defined by (12).
Output: T^* = optimal age replacement time;
$B(T^*)$ = optimal expected cost rate
Stop. End.

6. Special cases and concluding remarks

Our model is a generalization of the past models in the literature. We present some special cases in this section.

Case 1. $\overline{P}_0 = 1; \overline{P}_k = 0$ for $k = 1, 2, 3, \ldots$; $r_k(t) = r(t)$ for $k = 0, 1, 2, 3, \ldots$; $\phi(C(y), c_i(y)) = 0$. Using (12), we obtain

$$B(T) = \frac{\zeta_{01}(T)}{\int_0^T \overline{H}(t) \cdot G(t)\mathrm{d}t + \mu}, \tag{17}$$

where

$$\zeta_{01}(T) = c_1 \cdot \overline{H}(T) \cdot G(T) + c_2 \cdot \int_T^\infty \overline{H}(t)\mathrm{d}G(t) + c_3 \cdot \int_0^T G(t)\mathrm{d}H(t) + c_4 \cdot \int_0^\infty H(t)\mathrm{d}G(t) + c_h \cdot \int_0^T \overline{H}(t) \cdot G(t)\mathrm{d}t + c_s \cdot \int_0^\infty \overline{G}(t) \cdot H(t)\mathrm{d}t.$$

This expression agrees with (7) in Osaki and Yamada [23].

Case 2. $\overline{P}_0 = 1;\ \overline{P}_k = 0$ for $k = 1, 2, 3, \ldots;\ r_k(t) = r(t)$ for $k = 0, 1, 2, 3, \ldots;\ \phi(C(y), c_i(y)) = 0;\ c_h = 0;\ c_s = 0$. It follows from (12) that

$$B(T) = \frac{\zeta_{02}(T)}{\int_0^T \overline{H}(t) \cdot G(t)\mathrm{d}t + \mu}, \tag{18}$$

where

$$\zeta_{02}(T) = c_1 \cdot \overline{H}(T) \cdot G(T) + c_2 \cdot \int_T^\infty \overline{H}(t)\mathrm{d}G(t) + c_3 \cdot \int_0^T G(t)\mathrm{d}H(t) + c_4 \cdot \int_0^\infty H(t)\mathrm{d}G(t).$$

(18) agrees with the results in Nakagawa and Osaki [21].

Case 3. $\overline{P}_k = q^k$ for $k = 1, 2, 3, \ldots$; $r_k(t) = r(t)$ for $k = 0, 1, 2, 3, \ldots$; $\phi(C(y), c_i(y)) = C$; $c_h = 0$; $c_s = 0$; $G(t)$ is degenerated at 0. This is the case considered by Clèroux et al. [31]. Here we treat C as a truncated random variable with p.d.f. $w(x)/q$ for $0 \leqslant x \leqslant \xi$ and $q = \int_0^\xi w(x)\mathrm{d}x$. If we use $\overline{P}_k = q^k$ for $k = 1, 2, 3, \ldots$, $r_k(t) = r(t)$ for $k = 0, 1, 2, 3, \ldots$, $\alpha_i(t) = E_{C(t)}[\phi(C(t), c_i(t))] = E(C) = \left(\int_0^\xi x \cdot w(x)\mathrm{d}x\right)/q$, $c_h = 0$, $c_s = 0$ and $G(t) = 1$ in (12), then we obtain the same average cost function as in Clèroux et al. [31]:

$$B(T) = \frac{c_1 + \left[(c_3 - c_2) + \frac{\int_0^\xi x \cdot w(x)\mathrm{d}x}{p}\right]\left[1 - e^{-p\Lambda(T)}\right]}{\int_0^T e^{-p\Lambda(t)}\mathrm{d}t}, \tag{19}$$

where $p = 1 - q$ and $\Lambda(T) = \int_0^T r(u)\mathrm{d}u$.

Case 4. $\overline{P}_0 = 1$; $\overline{P}_k = 0$ for $k = 1, 2, 3, \ldots$; $r_k(t) = r(t)$ for $k = 0, 1, 2, 3, \ldots$; $\phi(C(y), c_i(y)) = 0$; $c_h = 0$; $c_s = 0$; $G(t)$ is degenerated at 0. From (12), we obtain

$$B(T) = \frac{c_1 + (c_3 - c_2)(1 - e^{-\Lambda(T)})}{\int_0^T e^{-\Lambda(t)}\mathrm{d}t}, \tag{20}$$

which is the same as the result in Barlow and Hunter [1].

Case 5. $\overline{P}_k = 1$ for $k = 0, 1, 2, 3, \ldots$; $r_k(t) = r(t)$ for $k = 0, 1, 2, 3, \ldots$; $\phi(C(y), c_i(y)) = c(t)$; $c_h = 0$; $c_s = 0$; $G(t)$ is degenerated at 0. (12) is reduced to

$$B(T) = \frac{c_1 + \int_0^T c(t) \cdot r(t)\mathrm{d}t}{T}, \tag{21}$$

which is the same as that in Boland [32].

Case 6. $\overline{P}_k = 1$ for $k = 0,1,2,3,\ldots$; $r_k(t) = r(t)$ for $k = 0,1,2,3,\ldots$; $\phi(C(y), c_i(y)) = c + c(t)$; $c_h = 0$; $c_s = 0$; $G(t)$ is degenerated at 0. Using (12), we have

$$B(T) = \frac{c_1 + c\cdot\Lambda(T) + \int_0^T c(t)\cdot r(t)\mathrm{d}t}{T}, \tag{22}$$

as shown in Tilquin and Clèroux [33].

Case 7. $\overline{P}_k = 1$ for $k = 0,1,2,3,\ldots$; $r_k(t) = r(t)$ for $k = 0,1,2,3,\ldots$; $\phi(C(y), c_i(y)) = c$; $c_h = 0$; $c_s = 0$; $G(t)$ is degenerated at 0. This is classical result

$$B(T) = \frac{c_1 + c\cdot\Lambda(T)}{T}, \tag{23}$$

obtained by Barlow and Hunter [1].

Case 8. $\overline{P}_k = 1$ for $k = 0,1,2,3,\ldots$; $r_k(t) = r(t)$ for $k = 0,1,2,3,\ldots$; $\phi(C(y), c_i(y)) = c_i$; $c_h = 0$; $c_s = 0$; $G(t)$ is degenerated at 0. This case becomes the model in Boland and Proschan [34] if $c_i = a + i\cdot c$ is utilized.

Case 9. $r_k(t) = r(t)$ for $k = 0,1,2,3,\ldots$. Using $\Lambda(t) = \int_0^t r(u)\mathrm{d}u$, and $P_k(t) = e^{-\Lambda(t)}\Lambda(t)^k/k!$ in (2), we get

$$B(T) = \frac{\zeta_8(T)}{\int_0^T \overline{H}(t)\cdot G(t)\mathrm{d}t + \mu}, \tag{23}$$

where

$$\begin{aligned}\zeta_8(T) = {} & c_1\cdot\overline{H}(T)\cdot G(T) + c_2\cdot\int_T^\infty \overline{H}(t)\mathrm{d}G(t) + c_3\cdot\int_0^T G(t)\mathrm{d}H(t) + c_4\cdot\int_0^\infty H(t)\mathrm{d}G(t)\\ & + \int_0^T \sum_{i=1}^\infty \alpha_i(t)\frac{e^{-\Lambda(t)}\Lambda(t)^{i-1}}{(i-1)!}r(t)\overline{P}_i G(t)\mathrm{d}t + \int_0^\infty\int_0^t \sum_{i=1}^\infty \alpha_i(y)\frac{e^{-\Lambda(t)}\Lambda(t)^{i-1}}{(i-1)!}r(y)\overline{P}_i\mathrm{d}y\mathrm{d}G(t)\\ & + c_h\cdot\int_0^T \overline{H}(t)\cdot G(t)\mathrm{d}t + c_s\cdot\int_0^\infty \overline{G}(t)\cdot H(t)\mathrm{d}t.\end{aligned}$$

This is in agreement with the results in Sheu and Chien [10].

It has been shown from these special cases that our model is more flexible in studying the maintenance policy of systems subject to a more general shock process and cost structure. Such a general model provides practitioners a useful tool in designing the optimal maintenance policy. However, it is assumed that the repair and replacement times are zero in this paper. Studying the maintenance policy for the system with non-zero repair and replacement times can be a direction of the future research.

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Appendix A. Proof of lemma 1

Let $M(t)$ be the number of type I failures over time interval $[0,t)$, then

$$\Phi(t) = \sum_{i=1}^{M(t)}\phi(C(Z_i), c_i(Z_i)),$$

where Z_i be the arrival time of ith type I failure given that no type II failure occurs in Z_i. Hence,

$$\begin{aligned}E[\Phi(t)] &= E\left[\sum_{i=1}^{M(t)}\phi(C(Z_i), c_i(Z_i))\right] = \sum_{i=1}^\infty E\left[\phi(C(Z_i), c_i(Z_i))\cdot I_{[0,t)}(Z_i)\right] = \sum_{i=1}^\infty\int_0^t E_{C(y)}[\phi(C(y), c_i(y))]\cdot P_r(Z_i\in(y, y+\mathrm{d}y))\\ &= \sum_{i=1}^\infty\int_0^t \alpha_i(y)\frac{P_r(S_i\in\mathrm{d}y, M>i)}{P_r(Y>y)} = \sum_{i=1}^\infty\int_0^t\alpha_i(y)\cdot\frac{\overline{P}_i}{\overline{H}(y)}\cdot P_r(S_i\in\mathrm{d}y),\Bigg]\end{aligned}$$

where S_i be the arrival time of ith shock for $i = 1,2,3,\ldots$, and

$$I_{[0,t)}(Z_i) = \begin{cases}1 & \text{if}\quad Z_i\in[0,t),\\ 0 & \text{otherwise}.\end{cases}$$

But

$$P_r\{S_i\leqslant y\} = P_r\{N(y)\geqslant i\} = \sum_{k=i}^\infty P_k(y)$$

and also by (3),

$$P_r\{S_i\in \mathrm{d}y\}=\frac{\mathrm{d}P_r\{S_i\leqslant y\}}{\mathrm{d}y}\mathrm{d}y=P_{i-1}(y)r_{i-1}(y)\mathrm{d}y.$$

Therefore, we have

$$E[\Phi(t)]=\int_0^t\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\cdot\overline{P}_i\cdot\frac{1}{\overline{H}(y)}\mathrm{d}y.$$

Appendix B. Derivation of (11)

$$\begin{aligned}
E(V_1) &= c_1\cdot\overline{H}(T)\cdot G(T)+E[\Phi(T)]\cdot\overline{H}(T)\cdot G(T)+c_h\cdot\overline{H}(T)\cdot\textstyle\int_0^T(T-t)\mathrm{d}G(t)\\
&\quad +c_2\cdot\textstyle\int_T^\infty\overline{H}(t)\mathrm{d}G(t)+\int_T^\infty E[\Phi(t)]\cdot\overline{H}(t)\mathrm{d}G(t)+c_3\cdot\int_0^T G(t)\mathrm{d}H(t)\\
&\quad +\textstyle\int_0^T E[\Phi(t)]\cdot G(t)\mathrm{d}H(t)+c_h\cdot\int_0^T\int_0^y(y-t)\mathrm{d}G(t)\mathrm{d}H(y)+c_4\cdot\int_0^\infty H(t)\mathrm{d}G(t)\\
&\quad +\textstyle\int_0^\infty\int_0^t E[\Phi(y)]\mathrm{d}H(y)\mathrm{d}G(t)+c_s\cdot\int_0^\infty\int_0^t(t-y)\mathrm{d}H(y)\mathrm{d}G(t)\\
&= c_1\cdot\overline{H}(T)\cdot G(T)+E[\Phi(T)]\cdot\overline{H}(T)\cdot G(T)+c_h\cdot\overline{H}(T)\cdot\textstyle\int_0^T(T-t)\mathrm{d}G(t)\\
&\quad +c_2\cdot\textstyle\int_T^\infty\overline{H}(t)\mathrm{d}G(t)+\int_T^\infty E[\Phi(t)]\cdot\overline{H}(t)\mathrm{d}G(t)+c_3\cdot\int_0^T G(t)\mathrm{d}H(t)\\
&\quad -E[\Phi(T)]\cdot G(T)\cdot\overline{H}(T)+\textstyle\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}E[\Phi(t)]+\int_0^T E[\Phi(t)]\cdot\overline{H}(t)\mathrm{d}G(t)\\
&\quad -c_h\cdot\overline{H}(T)\cdot\textstyle\int_0^T(T-t)\mathrm{d}G(t)+c_h\cdot\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t+c_4\cdot\int_0^\infty H(t)\mathrm{d}G(t)\\
&\quad -\textstyle\int_0^\infty E[\Phi(t)]\cdot\overline{H}(t)\mathrm{d}G(t)+\int_0^\infty\int_0^t\overline{H}(y)\mathrm{d}E[\Phi(y)]\mathrm{d}G(t)+c_s\cdot\int_0^\infty\overline{G}(t)\cdot H(t)\mathrm{d}t\\
&= c_1\cdot\overline{H}(T)\cdot G(T)+c_2\cdot\textstyle\int_T^\infty\overline{H}(t)\mathrm{d}G(t)+c_3\cdot\int_0^T G(t)\mathrm{d}H(t)+c_4\cdot\int_0^\infty H(t)\mathrm{d}G(t)\\
&\quad +\textstyle\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}E[\Phi(t)]+\int_0^\infty\int_0^t\overline{H}(y)\mathrm{d}E[\Phi(y)]\mathrm{d}G(t)\\
&\quad +c_h\cdot\textstyle\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t+c_s\cdot\int_0^\infty\overline{G}(t)\cdot H(t)\mathrm{d}t\\
&= c_1\cdot\overline{H}(T)\cdot G(T)+c_2\cdot\textstyle\int_T^\infty\overline{H}(t)\mathrm{d}G(t)+c_3\cdot\int_0^T G(t)\mathrm{d}H(t)+c_4\cdot\int_0^\infty H(t)\mathrm{d}G(t)\\
&\quad +\textstyle\int_0^\infty\sum_{i=1}^{\infty}\alpha_i(t)P_{i-1}(t)r_{i-1}(t)\overline{P}_iG(t)\mathrm{d}t+\int_0^\infty\int_0^t\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\overline{P}_i\mathrm{d}y\mathrm{d}G(t)\\
&\quad +c_h\cdot\textstyle\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t+c_s\cdot\int_0^\infty\overline{G}(t)\cdot H(t)\mathrm{d}t.
\end{aligned}$$

Appendix C. Proof of Theorem 1

$$\frac{d}{dT}B(T)=0$$

implies (14).

$$\begin{aligned}
Q(T)\equiv&\left[(c_3-c_2)r_H(T)+\left(\sum_{i=1}^{\infty}\alpha_i(T)P_{i-1}(T)r_{i-1}(T)\overline{P}_i\right)\frac{1}{\overline{H}(T)}+c_h\right]\left[\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t+\int_0^\infty\overline{G}(t)\mathrm{d}t\right]-(c_3-c_2)\cdot\int_0^T G(t)\mathrm{d}H(t)\\
&-\int_0^T\sum_{i=1}^{\infty}\alpha_i(t)P_{i-1}(t)r_{i-1}(t)\overline{P}_iG(t)\mathrm{d}t-c_h\cdot\int_0^T\overline{H}(t)\cdot G(t)\mathrm{d}t
\end{aligned}$$

$Q(T)$ is strictly increasing because of the assumption that $\varepsilon(T)$ is strictly increasing.

$$Q(0)=\varepsilon(0)\cdot\int_0^\infty\overline{G}(t)\mathrm{d}t,$$

$$\begin{aligned}
Q(\infty)=&\ \varepsilon(\infty)\cdot\left[\int_0^\infty\overline{H}(t)\cdot G(t)\mathrm{d}t+\int_0^\infty\overline{G}(t)\mathrm{d}t\right]-(c_3-c_2)\cdot\int_0^\infty G(t)\mathrm{d}H(t)-\int_0^\infty\sum_{i=1}^{\infty}\alpha_i(t)P_{i-1}(t)r_{i-1}(t)\overline{P}_iG(t)\mathrm{d}t\\
&-c_h\cdot\int_0^\infty\overline{H}(t)\cdot G(t)\mathrm{d}t,
\end{aligned}$$

$$K\equiv c_2\cdot\int_0^\infty G(t)\mathrm{d}H(t)+c_4\cdot\int_0^\infty H(t)\mathrm{d}G(t)+\int_0^\infty\int_0^t\sum_{i=1}^{\infty}\alpha_i(y)P_{i-1}(y)r_{i-1}(y)\overline{P}_i\mathrm{d}y\mathrm{d}G(y)+c_s\cdot\int_0^\infty\overline{G}(t)\cdot H(t)dt.$$

If $\varepsilon(0) < \delta$ and $\varepsilon(\infty) > \vartheta$, then $Q(0) < K$ and $Q(\infty) > K$. Thus from the strictly increasing property of $Q(T)$, there exists a unique and finite T^*, $0 < T^* < \infty$ satisfying (14), which minimizes $B(T)$. If T^* is the solution, then from (12) and (13), $B(T^*) = \varepsilon(T^*)$.

If $\varepsilon(\infty) \leqslant \vartheta$, then $Q(\infty) \leqslant K$. Thus $Q(T) < K$ for any finite T, which implies $B'(T) < 0$ for any finite T. Thus $T^* = \infty$, meaning that no planned replacement is needed.

If $\varepsilon(0) \geqslant \delta$, then $Q(0) \geqslant K$. Thus $Q(T) > K$ and $B'(T) > 0$ for any $T > 0$, which implies that $T^* = 0$ because $B(T)$ is strictly increasing in T.

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