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Embedding a Hamiltonian cycle in the crossed cube with two required vertices in the fixed positions ☆

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ABSTRACT

A Hamiltonian graph G is said to be panpositionably Hamiltonian if, for any two distinct vertices x and y of G, there is a Hamiltonian cycle C of G having $d_C(x,y) = l$ for any integer l satisfying $d_G(x,y) \leqslant l \leqslant \frac{|V(G)|}{2}$, where $d_G(x,y)$ (respectively, $d_C(x,y)$) denotes the distance between vertices x and y in G (respectively, C), and $|V(G)|$ denotes the total number of vertices of G. As the importance of Hamiltonian properties for data communication among units in an interconnected system, the panpositionable Hamiltonicity involves more flexible message transmission. In this paper, we study this property with respect to the class of crossed cubes, which is a popular variant of the hypercube network.

1. Introduction

In many parallel and distributed computer systems, processors are connected based on interconnection networks, so the interconnection network is a critical factor that affects system performance and it is widely addressed by researchers [14,20,26]. In terms of network analysis, the topological structure of an interconnection network can be modeled as a graph whose vertices and edges represent processors and communication links, respectively. Among many kinds of network topologies, the binary n-cube [23] (for short, hypercube) is one of the most popular networks for parallel and distributed computation. Not only is it ideally suited to both special-purpose and general-purpose tasks, but it can efficiently simulate many other networks [14,20,26]. However, the hypercube is bipartite so that it cannot make the best use of its hardware resources. For example, the hypercube has the largest diameter among cube family. To compensate for this drawback, many researchers [1,7,8,28] try to fashion networks with lower diameters. One such network topology is the crossed cube, which was first proposed by Efe [9]. The crossed cube is derived from the hypercube by changing the connection of some links. Its diameter is about half of the hypercube's [6,9]. Besides, the crossed cube has many attractive properties. For example, it has more cycles than the hypercube [12], and binary trees can be embedded into it [18]. Moreover, the embedding of paths of odd and even lengths [11,13] and the embedding of many-to-many disjoint path covers [22] can be done in the crossed cube. The definition of the crossed cube will be presented in the next section.

Throughout this paper, graphs are finite, simple, and undirected. Some important graph-theoretic definitions and notations will be introduced in advance. For those not defined here, however, we follow the standard terminology given by Bondy and Murty [4]. An *undirected graph* G is a graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| > 0$ and $E(G) \subseteq \{(u,v) | (u,v)$ is an unordered pair of $V(G)\}$. Two vertices u and v of G are *adjacent* if $(u,v) \in E(G)$. The *degree* of a vertex

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u in G is the number of edges incident to u. A graph G is *k-regular* if all its vertices have the same degree k. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; H is a *spanning* subgraph of G (equivalently, H *spans* G) if $V(H) = V(G)$. Let S be a nonempty subset of $V(G)$. The subgraph of G *induced* by S is a graph whose vertex set is S and whose edge set consists of all the edges of G joining any two vertices in S.

A *path* P of length k, $k \geqslant 1$, from vertex x to vertex y in G is a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_{k+1}\rangle$ such that $v_1 = x$, $v_{k+1} = y$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leqslant i \leqslant k$. Moreover, a path of length 0, consisting of a single vertex x, is denoted by $\langle x\rangle$. We can write P as $\langle v_1, v_2, \ldots, v_i, Q, v_j, \ldots, v_{k+1}\rangle$ for convenience if we know that $Q = \langle v_i, \ldots, v_j\rangle$, where $i \leqslant j$. The ith vertex of P is denoted by $P(i)$; i.e., $P(i) = v_i$. In particular, let $rev(P)$ represent the reverse of P; that is, $rev(P) = \langle v_{k+1}, v_k, \ldots, v_1\rangle$. We use $\ell(P)$ to denote the *length* of P. The *distance* between two distinct vertices u and v in G, denoted by $d_G(u, v)$, is the length of the shortest path between u and v. A *cycle* is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length k, $k \geqslant 3$, is represented by $\langle v_1, v_2, \ldots, v_k, v_1\rangle$. A path (or cycle) is a *Hamiltonian path* (or *Hamiltonian cycle*) of G if it spans G. A graph G is *Hamiltonian* if it has a Hamiltonian cycle, and a graph G is *Hamiltonian connected* if it contains a Hamiltonian path joining any pair of distinct vertices.

In recent years, many research results about cycle embedding have been focused on exploring the properties of pancyclicity [5,12,17,21,27]. A graph G is called *pancyclic* [3] if it contains a cycle of length l for each integer l from 3 to $|V(G)|$ inclusive. More specifically, a graph G is called *edge-pancyclic* (respectively, *vertex-pancyclic*) if its any edge (respectively, vertex) lies on a cycle of length l for every $3 \leqslant l \leqslant |V(G)|$. On the other hand, graph G is said to be *panconnected* [2] if, for any two distinct vertices x and y, it has a path of length l joining x and y for any integer l satisfying $d_G(x,y) \leqslant l \leqslant |V(G)| - 1$. It is easy to see that every panconnected graph must be pancyclic, edge-pancyclic, and vertex-pancyclic. In order to have insight into the topological properties related to cycle embedding, Kao et al. [16] paid attention to embedding Hamiltonian cycles and proposed an intriguing property as follows: A graph G is *panpositionably Hamiltonian* [16] if, for any two distinct vertices x and y of G, there is a Hamiltonian cycle C of G having $d_C(x,y) = l$ for any integer l satisfying $d_G(x,y) \leqslant l \leqslant \frac{|V(G)|}{2}$. Such a concept is called the *panpositionable Hamiltonicity*. As the significance of Hamiltonian properties for data communication among processors/computers in an interconnected system, the panpositionable Hamiltonicity involves much more flexible message transmission. Later, Teng et al. [24,25] studied the panpositionable Hamiltonicity with respect to alternating group graphs and arrangement graphs, and they gave an example to show that a panconnected graph is not necessarily panpositionably Hamiltonian [24].

In this paper, we investigate the panpositionable Hamiltonicity for the class of crossed cubes. The rest of this paper is organized as follows. In Section 2, the definition of crossed cubes is introduced. In Section 3, the proofs of two main theorems are given. Finally, some concluding remarks are given in Section 4.

2. The crossed cube and its properties

The n-dimensional crossed cube, denoted by CQ_n, has 2^n vertices, each of which corresponds to an n-bit binary string. To define the crossed cube, an additional concept "pair related" has to be introduced first.

Definition 1. Two 2-bit binary strings $\mathbf{x} = x_2x_1$ and $\mathbf{y} = y_2y_1$ are pair related, denoted by $\mathbf{x} \sim \mathbf{y}$, if and only if $(\mathbf{x},\mathbf{y}) \in \{(00,00), (10,10), (01,11), (11,01)\}$.

The definition of CQ_n is given as below.

Definition 2. The n-dimensional crossed cube CQ_n is recursively constructed as follows: (i) CQ_1 is a complete graph with two vertices, whose vertex set is represented by $\{0,1\}$. (ii) For $n \geqslant 2$, let CQ^0_{n-1} and CQ^1_{n-1} be two copies of CQ_{n-1} with $V\left(CQ^0_{n-1}\right) = \{0u_{n-1}u_{n-2}\cdots u_1 | u_i = 0 \text{ or } 1 \text{ for } 1 \leqslant i \leqslant n-1\}$ and $V(CQ^1_{n-1}) = \{1u_{n-1}u_{n-2}\cdots u_1 | u_i = 0 \text{ or } 1 \text{ for } 1 \leqslant i \leqslant n-1\}$. Then, CQ_n is formed by connecting CQ^0_{n-1} and CQ^1_{n-1} with 2^{n-1} edges so that a vertex $\mathbf{u} = 0u_{n-1}u_{n-2}\cdots u_1$ in CQ^0_{n-1} is adjacent to a vertex $\mathbf{v} = 1v_{n-1}v_{n-2}\cdots v_1$ in CQ^1_{n-1} if and only if

(1) $u_{n-1} = v_{n-1}$ if n is even, and
(2) $u_{2i}u_{2i-1} \sim v_{2i}v_{2i-1}$ for all i, $1 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor$.

For the sake of convenience, we denote this recursive construction by $CQ_n = CQ^0_{n-1} \otimes CQ^1_{n-1}$. From the above definition, CQ_2 is just a cycle of length 4, and CQ_n is an n-regular graph. We depict CQ_3 and CQ_4 in Fig. 1. It is proved that CQ_n is n-connected [19] and has diameter $\lceil \frac{n+1}{2} \rceil$ [9]. Furthermore, the crossed cube receives many researchers' attention [6,10,13,15,27] since it was introduced.

In [9], Efe proposed a shortest path routing algorithm **Route**$(\mathbf{u},\mathbf{v})$ for CQ_n, which implies the following two lemmas.

Lemma 1 [9]. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be any two different vertices of* CQ_n *such that* $\{\mathbf{u},\mathbf{v}\} \subset V(CQ^i_{n-1})$, $i \in \{0,1\}$. *Then,* $d_{CQn}(\mathbf{u},\mathbf{v}) = d_{CQ^i_{n-1}}(\mathbf{u},\mathbf{v})$.

Lemma 2 [9]. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be any two vertices of* CQ_n, $n \geqslant 2$, *such that* $\mathbf{u}$ *is in* CQ^0_{n-1} *and* $\mathbf{v}$ *is in* CQ^1_{n-1}. *Suppose that* $\mathbf{x}$ *is the vertex in* CQ^1_{n-1} *adjacent to* $\mathbf{u}$, *and* $\mathbf{y}$ *is the vertex in* CQ^0_{n-1} *adjacent to* $\mathbf{v}$. *Then,* $d_{CQ_n}(\mathbf{u},\mathbf{v}) = d_{CQ_n}(\mathbf{u},\mathbf{y}) + 1$ *or* $d_{CQ_n}(\mathbf{u},\mathbf{v}) = d_{CQ_n}(\mathbf{x},\mathbf{v}) + 1$.

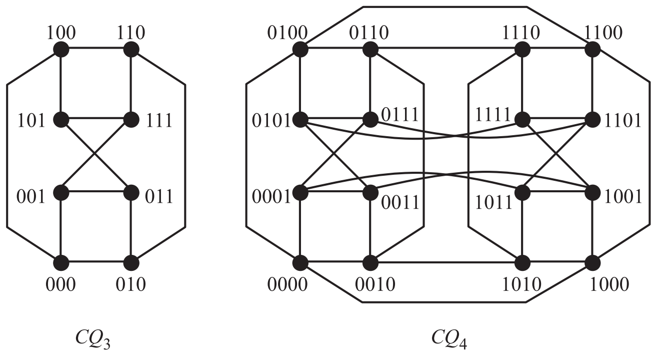

Fig. 1. Illustration of CQ_3 and CQ_4.

Fan et al. [13] proved that, for any two vertices **u** and **v** in CQ_n, there exists a path of length l joining **u** and **v** for any integer l with $d_{CQ_n}(\mathbf{u},\mathbf{v}) \leqslant l \leqslant 2^n - 1$ and $l \neq d_{CQ_n}(\mathbf{u},\mathbf{v}) + 1$.

A Hamiltonian graph G is said to be *f-fault-tolerant Hamiltonian* if $G - F$ remains Hamiltonian for every $F \subseteq V(G) \cup E(G)$ with $|F| \leqslant f$. A Hamiltonian connected graph G is said to be *f-fault-tolerant Hamiltonian connected* if $G - F$ remains Hamiltonian connected for every $F \subseteq V(G) \cup E(G)$ with $|F| \leqslant f$.

Lemma 3 [15]. *For any integer n, $n \geqslant 3$, CQ_n is $(n-2)$-fault-tolerant Hamiltonian and $(n-3)$-fault-tolerant Hamiltonian connected.*

Any vertex $\mathbf{u} = u_n u_{n-1} \cdots u_1$ in CQ_n is said to be adjacent to a vertex $\mathbf{v} = v_n v_{n-1} \cdots v_1$ along the ith dimension, $1 \leqslant i \leqslant n$, if the following four conditions are all satisfied: (i) $u_i \neq v_i$, (ii) $u_j = v_j$ for all j, $i+1 \leqslant j \leqslant n$, (iii) $u_{2k}u_{2k-1} \sim v_{2k}v_{2k-1}$ for all k, $1 \leqslant k \leqslant \lfloor \frac{i-1}{2} \rfloor$, and (iv) $u_{i-1} = v_{i-1}$ if i is even. Then, we say that **u** is the *i-neighbor* of **v**, denoted by $(\mathbf{v})^i$, and vice versa. The edge $(\mathbf{u},(\mathbf{u})^i)$ is called an *i-dimensional edge*. It is easy to see that $\mathbf{v} = (\mathbf{u})^i$ if and only if $\mathbf{u} = (\mathbf{v})^i$.

Lemma 4. *Let **u** be any vertex of CQ_n, $n \geqslant 3$. For any integer i, $1 \leqslant i \leqslant n-1$, $((\mathbf{u})^i)^n = ((\mathbf{u})^n)^i$ if (1) i is even, or (2) $i = n-1$ with n even.*

Proof. Let **u**, **v**, **x**, **y**, and **z** be five vertices of CQ_n, where $\mathbf{v} = (\mathbf{u})^n$, $\mathbf{x} = (\mathbf{u})^i$, $\mathbf{y} = (\mathbf{v})^i$, and $\mathbf{z} = (\mathbf{x})^n$. Since $CQ_n = CQ^0_{n-1} \otimes CQ^1_{n-1}$, we assume, without loss of generality, that $\mathbf{u} \in V(CQ^0_{n-1})$. According to the possible values of n and i, we distinguish the following three cases.

Case 1. n is even and $i = 2j$ for any j, $1 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor$. By the definition of i-neighbors, we can obtain immediately that $y_p = z_p$ for $1 \leqslant p \leqslant n$ except $p \in \{i, i-1\}$. Moreover, $z_i z_{i-1} \sim \bar{u}_i u_{i-1}$, $y_i y_{i-1} = \bar{v}_i v_{i-1}$, and $v_i v_{i-1} \sim u_i u_{i-1}$, where $\bar{u}_i$ denotes the complement of u_i. According to the possible values of u_{i-1}, we consider the following two subcases.

Subcase 1.1. $u_{i-1} = 0$. This results in $z_i z_{i-1} = \bar{u}_i 0$, $v_i v_{i-1} = u_i 0$, and $y_i y_{i-1} = \bar{v}_i v_{i-1} = \bar{u}_i 0$. It yields $\mathbf{y} = \mathbf{z}$.

Subcase 1.2. $u_{i-1} = 1$. This results in $z_i z_{i-1} = u_i 1$, $v_i v_{i-1} = \bar{u}_i 1$, and $y_i y_{i-1} = \bar{v}_i v_{i-1} = u_i 1$. This also yields $\mathbf{y} = \mathbf{z}$. Thus, $((\mathbf{u})^i)^n = ((\mathbf{u})^n)^i$ when n is even and $i = 2j$ for any j, $1 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor$.

Case 2. n is even and $i = n-1$. In this case, by the definition of i-neighbors, we can derive directly that $y_p = z_p$ for $1 \leqslant p \leqslant n-2$. Furthermore, $y_n y_{n-1} = v_n \bar{v}_{n-1} = \bar{u}_n \bar{u}_{n-1}$ and $z_n z_{n-1} = \bar{x}_n x_{n-1} = \bar{u}_n \bar{u}_{n-1}$. This reveals that $\mathbf{y} = \mathbf{z}$.

Case 3. n is odd. By a similar argument as that in Case 1, this case holds. This concludes the proof of this lemma. □

Corollary 1. *Let $(\mathbf{u},\mathbf{v})$ be any n-dimensional edge in CQ_n, $n \geqslant 3$. For any integer i, $1 \leqslant i \leqslant n-1$, the set of vertices $\{\mathbf{u},\mathbf{v},(\mathbf{u})^i,(\mathbf{v})^i\}$ induces a cycle of length 4 if (1) i is even, or (2) $i = n-1$ with n even.*

In [12], Fan et al. introduced how to locate a cycle of length 5 as the following lemma.

Lemma 5 [12]. *Let $(\mathbf{u},\mathbf{v})$ be any n-dimensional edge in CQ_n, $n \geqslant 3$. Then, $((\mathbf{u})^1)^n = ((\mathbf{v})^2)^1 = ((\mathbf{v})^1)^2$.*

(i) The set of vertices $\{\mathbf{u},\mathbf{v},(\mathbf{u})^1,(\mathbf{v})^2,((\mathbf{v})^2)^1\}$ *induces a cycle of length 5.*
(ii) The set of vertices $\{\mathbf{u},\mathbf{v},(\mathbf{u})^1,(\mathbf{v})^1,((\mathbf{v})^1)^2\}$ *induces a cycle of length 5.*

The following lemma describes a good property of CQ_4. It can be verified by brute force with a computer program [29].

Lemma 6. *Let* $(\mathbf{x},\mathbf{y})$ *be any 2-dimensional, 3-dimensional, or 4-dimensional edge of* CQ_4*. Then,* $CQ_4-\{\mathbf{x},\mathbf{y}\}$ *has a Hamiltonian path between two arbitrary vertices.*

Corollary 2 is drawn from Lemmas 3 and 6.

Corollary 2. *For* $n\geqslant 4$ *and* $2\leqslant i\leqslant n$*, let* $(\mathbf{x},\mathbf{y})$ *be any i-dimensional edge of* CQ_n*. Then,* $CQ_n-\{\mathbf{x},\mathbf{y}\}$ *has a Hamiltonian path between two arbitrary vertices.*

3. Panpositionable Hamiltonicity of CQ_n

By the definition, CQ_n has no cycle of length 3 as a subgraph so that there does not exist any path of length 2 between adjacent vertices $\mathbf{x}$ and $\mathbf{y}$ in CQ_n. For this reason, CQ_n is not panpositionably Hamiltonian. However, Theorem 2 will show a relaxed version of the panpositionable Hamiltonicity. Before introducing Theorem 2, we prove the following theorem in advance.

Theorem 1. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be any two vertices of* CQ_n*,* $n\geqslant 4$*, and let l be any integer with* $d_{CQ_n}(\mathbf{x},\mathbf{y})\leqslant l\leqslant 2^n-1$ *and* $l\neq d_{CQ_n}(\mathbf{x},\mathbf{y})+1$*. There exists a Hamiltonian path P of* CQ_n *such that* $P(1)=\mathbf{x}$ *and* $P(l+1)=\mathbf{y}$*.*

Proof. We prove this theorem by induction on n. Firstly, the correctness of the induction base on CQ_4 can be verified by brute force with a computer program [30]. The inductive hypothesis is that the statement holds for any CQ_k, $4\leqslant k\leqslant n-1$. Then, we need to show that CQ_n has a Hamiltonian path P such that $P(1)=\mathbf{x}$ and $P(l+1)=\mathbf{y}$. Since $CQ_n=CQ^0_{n-1}\otimes CQ^1_{n-1}$, we assume, without loss of generality, that $\mathbf{x}\in V(CQ^0_{n-1})$. The following three cases are distinguished.

Case 1. $\mathbf{y}\in V(CQ^0_{n-1})$. By Lemma 1, we have $d_{CQ_n}(\mathbf{x},\mathbf{y})=d_{CQ^0_{n-1}}(\mathbf{x},\mathbf{y})$. The following three subcases have to be considered.

Subcase 1.1. $d_{CQ_n}(\mathbf{x},\mathbf{y})\leqslant l\leqslant 2^{n-1}-1$ and $l\neq d_{CQ_n}(\mathbf{x},\mathbf{y})+1$. By the inductive hypothesis, there exists a Hamiltonian path R of CQ^0_{n-1} such that $R(1)=\mathbf{x}$ and $R(l+1)=\mathbf{y}$. For convenience, path R is written as $\langle\mathbf{x},R_1,\mathbf{y},R_2,\mathbf{z}\rangle$, where $\mathbf{z}$ is some vertex in CQ^0_{n-1}. It is noticed that $\mathbf{z}=\mathbf{y}$ if $l=2^{n-1}-1$. By Lemma 3, we can find a Hamiltonian path S of CQ^1_{n-1} joining $(\mathbf{z})^n$ to any vertex $\mathbf{a}$ in CQ^1_{n-1}. Then, $P=\langle\mathbf{x},R_1,\mathbf{y},R_2,\mathbf{z},(\mathbf{z})^n,S,\mathbf{a}\rangle$ is a Hamiltonian path of CQ_n with $P(1)=\mathbf{x}$ and $P(l+1)=\mathbf{y}$, see Fig. 2(a) for illustration.

Subcase 1.2. $l=2^{n-1}$. Let $\mathbf{a}$ be any vertex in CQ^1_{n-1} other than $(\mathbf{x})^n$ and $(\mathbf{y})^n$. By Lemma 3, there exists a Hamiltonian path S of $CQ^1_{n-1}-\{\mathbf{a}\}$ joining $(\mathbf{x})^n$ to $(\mathbf{y})^n$. Similarly, there exists a Hamiltonian path R of $CQ^0_{n-1}-\{\mathbf{x}\}$ joining $(\mathbf{a})^n$ and $\mathbf{y}$. Then, $P=\langle\mathbf{x},(\mathbf{x})^n,S,(\mathbf{y})^n,\mathbf{y},R,(\mathbf{a})^n,\mathbf{a}\rangle$ is a Hamiltonian path of CQ_n such that $P(1)=\mathbf{x}$ and $P(2^{n-1}+1)=\mathbf{y}$. Fig. 2(b) illustrates this subcase.

Subcase 1.3. $2^{n-1}+1\leqslant l\leqslant 2^n-1$. By Lemma 3, there exists a Hamiltonian path R of CQ^0_{n-1} joining $\mathbf{x}$ to $\mathbf{y}$. For clarity, path R is written as $\langle\mathbf{x},R_1,\mathbf{a},\mathbf{b},R_2,\mathbf{y}\rangle$, where $\mathbf{a}$ and $\mathbf{b}$ are adjacent vertices satisfying $\ell(R_1)=l-2^{n-1}-1$. It is noticed that $\mathbf{x}=\mathbf{a}$ if $l=2^{n-1}+1$, and $\mathbf{b}=\mathbf{y}$ if $l=2^n-1$. Again, Lemma 3 ensures that CQ^1_{n-1} has a Hamiltonian path S joining $(\mathbf{a})^n$ to $(\mathbf{y})^n$. Then, $P=\langle\mathbf{x},R_1,\mathbf{a},(\mathbf{a})^n,S,(\mathbf{y})^n,\mathbf{y},rev(R_2),\mathbf{b}\rangle$ is a Hamiltonian path of CQ_n such that $P(1)=\mathbf{x}$ and $P(l+1)=\mathbf{y}$, see Fig. 2(c) for illustration.

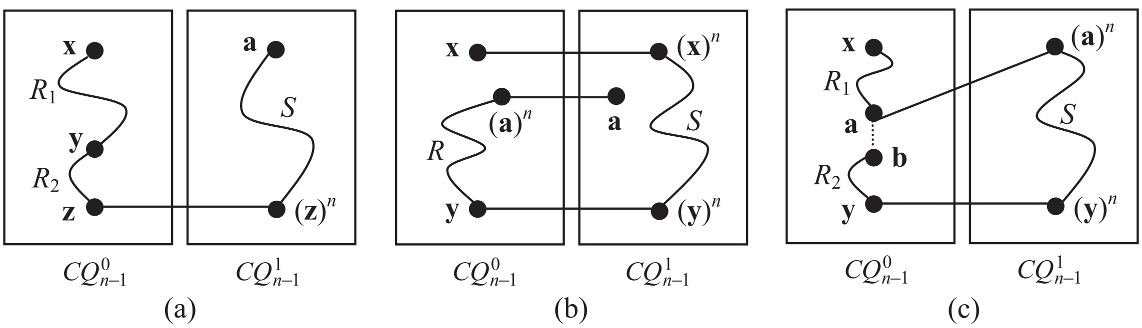

Fig. 2. Case 1 in the Proof of Theorem 1. (A dashed line or a straight line represents an edge.)

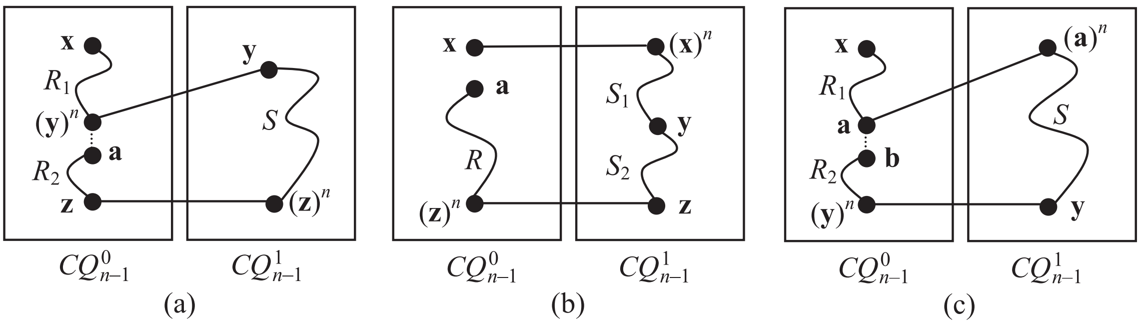

Fig. 3. Case 2 in the Proof of Theorem 1. (A dashed line or a straight line represents an edge.)

Case 2. $\mathbf{y} \in V(CQ^1_{n-1})$ and $(\mathbf{x},\mathbf{y}) \notin E(CQ_n)$. The following three subcases are distinguished.

Subcase 2.1. $d_{CQ_n}(\mathbf{x},\mathbf{y}) \leqslant l \leqslant 2^{n-1} - 1$ and $l \neq d_{CQ_n}(\mathbf{x},\mathbf{y}) + 1$. By Lemmas 1 and 2, we have $d_{CQ^0_{n-1}}(\mathbf{x},(\mathbf{y})^n) = d_{CQ_n}(\mathbf{x},(\mathbf{y})^n) = d_{CQ_n}(\mathbf{x},\mathbf{y}) - 1$ or $d_{CQ^1_{n-1}}((\mathbf{x})^n,\mathbf{y}) = d_{CQ_n}((\mathbf{x})^n,\mathbf{y}) = d_{CQ_n}(\mathbf{x},\mathbf{y}) - 1$. Firstly, we assume that $d_{CQ^0_{n-1}}(\mathbf{x},(\mathbf{y})^n) = d_{CQ_n}(\mathbf{x},\mathbf{y}) - 1$. By the inductive hypothesis, there exists a Hamiltonian path R of CQ^0_{n-1} with $R(1) = \mathbf{x}$ and $R(l) = (\mathbf{y})^n$. For clarity, path R is written as $\langle \mathbf{x}, R_1, (\mathbf{y})^n, \mathbf{a}, R_2, \mathbf{z} \rangle$, where $\mathbf{a}$ and $\mathbf{z}$ are vertices in $V(CQ^0_{n-1}) - \{\mathbf{x},(\mathbf{y})^n\}$. It is noticed that $\mathbf{a} = \mathbf{z}$ if $l = 2^{n-1} - 1$. By Lemma 3, there exists a Hamiltonian path S of CQ^1_{n-1} joining $\mathbf{y}$ to $(\mathbf{z})^n$. Then, $P = \langle \mathbf{x}, R_1, (\mathbf{y})^n, \mathbf{y}, S, (\mathbf{z})^n, \mathbf{z}, rev(R_2), \mathbf{a} \rangle$ is a Hamiltonian path of CQ_n such that $P(1) = \mathbf{x}$ and $P(l+1) = \mathbf{y}$. This subcase is illustrated in Fig. 3(a).

Next, we assume that $d_{CQ^1_{n-1}}((\mathbf{x})^n,\mathbf{y}) = d_{CQ_n}(\mathbf{x},\mathbf{y}) - 1$. By the inductive hypothesis, there exists a Hamiltonian path S of CQ^1_{n-1} with $S(1) = (\mathbf{x})^n$ and $S(l) = \mathbf{y}$. The path S can be written as $\langle (\mathbf{x})^n, S_1, \mathbf{y}, S_2, \mathbf{z} \rangle$, where $\mathbf{z}$ is some vertex in $V(CQ^1_{n-1}) - \{(\mathbf{x})^n, \mathbf{y}\}$. By Lemma 3, there exists a Hamiltonian path R of $CQ^0_{n-1} - \{\mathbf{x}\}$ joining $(\mathbf{z})^n$ to any vertex $\mathbf{a}$ in $V(CQ^0_{n-1}) - \{\mathbf{x},(\mathbf{z})^n\}$. Then, $P = \langle \mathbf{x}, (\mathbf{x})^n, S_1, \mathbf{y}, S_2, \mathbf{z}, (\mathbf{z})^n, R, \mathbf{a} \rangle$ is a Hamiltonian path of CQ_n such that $P(1) = \mathbf{x}$ and $P(l+1) = \mathbf{y}$. See Fig. 3(b) for illustration.

Subcase 2.2. $2^{n-1} \leqslant l \leqslant 2^n - 2$. Let $h = l - 2^{n-1}$. By Lemma 3, CQ^0_{n-1} has a Hamiltonian path R joining $\mathbf{x}$ and $(\mathbf{y})^n$. For convenience, path R is written as $\langle \mathbf{x}, R_1, \mathbf{a}, \mathbf{b}, R_2, (\mathbf{y})^n \rangle$, where $\mathbf{a}$ and $\mathbf{b}$ are two adjacent vertices in CQ^0_{n-1} with $\ell(R_1) = h$. It is noticed that $\mathbf{a} = \mathbf{x}$ if $l = 2^{n-1}$ and $\mathbf{b} = (\mathbf{y})^n$ if $l = 2^n - 2$. Obviously, CQ^1_{n-1} has a Hamiltonian path S joining $(\mathbf{a})^n$ to $\mathbf{y}$. Then, $P = \langle \mathbf{x}, R_1, \mathbf{a}, (\mathbf{a})^n, S, \mathbf{y}, (\mathbf{y})^n, rev(R_2), \mathbf{b} \rangle$ is a Hamiltonian path of CQ_n with $P(1) = \mathbf{x}$ and $P(l+1) = \mathbf{y}$. See Fig. 3(c) for illustration.

Subcase 2.3. $l = 2^n - 1$. By Lemma 3, CQ_n is Hamiltonian connected. Thus, there exists a Hamiltonian path P of CQ_n joining $\mathbf{x}$ to $\mathbf{y}$.

Case 3. $\mathbf{y} \in V(CQ^1_{n-1})$ and $(\mathbf{x},\mathbf{y}) \in E(CQ_n)$. Since $(\mathbf{x},\mathbf{y}) \in E(CQ_n)$, we have $1 \leqslant l \leqslant 2^n - 1$ and $l \neq 2$. When $l = 1$, it follows from Lemma 3 that there exists a Hamiltonian path R of $CQ_n - \{\mathbf{x}\}$ joining $\mathbf{y}$ to any vertex $\mathbf{z}$, $\mathbf{z} \neq \mathbf{x}$, so that $P = \langle \mathbf{x}, \mathbf{y}, R, \mathbf{z} \rangle$ is our required path with $P(1) = \mathbf{x}$ and $P(2) = \mathbf{y}$. Thus, we discuss $3 \leqslant l \leqslant 2^n - 1$. The following subcases are distinguished.

Subcase 3.1. $l = 3$. By Corollary 1, the set of vertices $\{\mathbf{x}, \mathbf{y}, (\mathbf{x})^2, (\mathbf{y})^2\}$ induces a cycle of length 4. By Lemma 3, CQ_n is $(n-3)$-fault-tolerant Hamiltonian connected. Therefore, there exists a Hamiltonian path R of $CQ_n - \{(\mathbf{x})^2, (\mathbf{y})^2\}$ joining $\mathbf{y}$ to $\mathbf{x}$. We can write R as $\langle \mathbf{y}, R', \mathbf{z}, \mathbf{x} \rangle$, where $\mathbf{z}$ is some vertex in $V(CQ_n) - \{\mathbf{x}, \mathbf{y}, (\mathbf{x})^2, (\mathbf{y})^2\}$. Then, $P = \langle \mathbf{x}, (\mathbf{x})^2, (\mathbf{y})^2, \mathbf{y}, R', \mathbf{z} \rangle$ is a Hamiltonian path of CQ_n such that $P(1) = \mathbf{x}$ and $P(4) = \mathbf{y}$. Fig. 4(a) illustrates this subcase.

Subcase 3.2. $l = 4$. By Lemma 5, the set of vertices $\{\mathbf{x}, \mathbf{y}, (\mathbf{x})^1, (\mathbf{y})^1, ((\mathbf{y})^1)^2\}$ induces a cycle of length 5. Let $\mathbf{v}$ be any vertex in $V(CQ^1_{n-1}) - \{\mathbf{y}, (\mathbf{y})^1, ((\mathbf{y})^1)^2\}$. By Corollary 2, there exists a Hamiltonian path S of $CQ^1_{n-1} - \{(\mathbf{y})^1, ((\mathbf{y})^1)^2\}$ joining $\mathbf{y}$ to $\mathbf{v}$. By Lemma 3, there exists a Hamiltonian path R of $CQ^0_{n-1} - \{(\mathbf{x})^1\}$ joining $(\mathbf{v})^n$ to $\mathbf{x}$. Path R can be written as $\langle (\mathbf{v})^n, R', \mathbf{u}, \mathbf{x} \rangle$, where $\mathbf{u}$ is some vertex adjacent to $\mathbf{x}$. Therefore, $P = \langle \mathbf{x}, (\mathbf{x})^1, ((\mathbf{y})^1)^2, (\mathbf{y})^1, \mathbf{y}, S, \mathbf{v}, (\mathbf{v})^n, R', \mathbf{u} \rangle$ is a Hamiltonian path of CQ_n with $P(1) = \mathbf{x}$ and $P(5) = \mathbf{y}$. The illustration of this subcase is shown in Fig. 4(b).

Subcase 3.3. $5 \leqslant l \leqslant 2^{n-1}$. Let $h = l - 2$. Then, we have $3 \leqslant h \leqslant 2^{n-1} - 2$. By Corollary 1, the set of vertices $\{\mathbf{x}, \mathbf{y}, (\mathbf{x})^2, (\mathbf{y})^2\}$ induces a cycle of length 4. By the inductive hypothesis, there exists a Hamiltonian path R of CQ^0_{n-1} such that $R(1) = (\mathbf{x})^2$ and $R(h+1) = \mathbf{x}$. For clarity, path R is written as $\langle (\mathbf{x})^2, R_1, \mathbf{x}, \mathbf{v}, R_2, \mathbf{z} \rangle$, where $\mathbf{v}$ and $\mathbf{z}$ are two vertices in $V(CQ^0_{n-1}) - \{\mathbf{x}, (\mathbf{x})^2\}$. It is noticed that $\mathbf{v} = \mathbf{z}$ if $h = 2^{n-1} - 2$. By Lemma 3, there exists a Hamiltonian path S of $CQ^1_{n-1} - \{(\mathbf{y})^2\}$ joining $\mathbf{y}$ to $(\mathbf{z})^n$. Then, $P = \langle \mathbf{x}, rev(R_1), (\mathbf{x})^2, (\mathbf{y})^2, \mathbf{y}, S, (\mathbf{z})^n, \mathbf{z}, rev(R_2), \mathbf{v} \rangle$ is a Hamiltonian path of CQ_n with $P(1) = \mathbf{x}$ and $P(l+1) = \mathbf{y}$, see Fig. 4(c) for illustration.

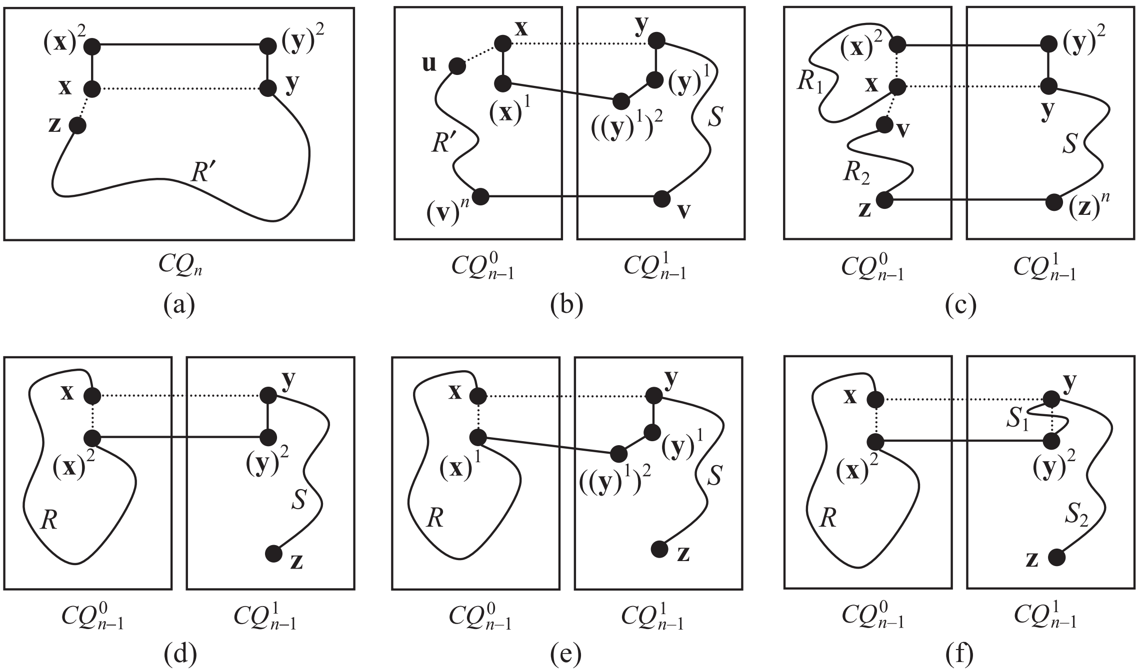

Fig. 4. Case 3 in the Proof of Theorem 1. (A dashed line or a straight line represents an edge.)

Subcase 3.4. $l = 2^{n-1} + 1$. Obviously, the set of vertices $\{\mathbf{x},\mathbf{y},(\mathbf{x})^2,(\mathbf{y})^2\}$ induces a cycle of length 4. By Lemma 3, there exists a Hamiltonian path R of CQ^0_{n-1} joining $\mathbf{x}$ and $(\mathbf{x})^2$. Similarly, there exists a Hamiltonian path S of $CQ^1_{n-1} - \{(\mathbf{y})^2\}$ joining $\mathbf{y}$ to any vertex $\mathbf{z}$ in $V(CQ^1_{n-1}) - \{(\mathbf{y})^2\}$. Then, $P = \langle \mathbf{x},R,(\mathbf{x})^2,(\mathbf{y})^2,\mathbf{y},S,\mathbf{z}\rangle$ is a Hamiltonian path of CQ_n with $P(1) = \mathbf{x}$ and $P(2^{n-1}+2) = \mathbf{y}$. Fig. 4(d) illustrates this subcase.

Subcase 3.5. $l = 2^{n-1} + 2$. By Lemma 5, the set of vertices $\{\mathbf{x},\mathbf{y},(\mathbf{x})^1,(\mathbf{y})^1,((\mathbf{y})^1)^2\}$ induces a cycle of length 5. By Lemma 3, CQ^0_{n-1} has a Hamiltonian path R joining $\mathbf{x}$ to $(\mathbf{x})^1$. By Corollary 2, $CQ^1_{n-1} - \{(\mathbf{y})^1,((\mathbf{y})^1)^2\}$ has a Hamiltonian path S joining $\mathbf{y}$ to any vertex $\mathbf{z}$ in $V(CQ^1_{n-1}) - \{\mathbf{y},(\mathbf{y})^1,((\mathbf{y})^1)^2\}$. Then, $P = \langle \mathbf{x},R,(\mathbf{x})^1,((\mathbf{y})^1)^2,(\mathbf{y})^1,\mathbf{y},S,\mathbf{z}\rangle$ is a Hamiltonian path of CQ_n with $P(1) = \mathbf{x}$ and $P(2^{n-1}+3) = \mathbf{y}$, see Fig. 4(e).

Subcase 3.6. $2^{n-1} + 3 \leqslant l \leqslant 2^n - 1$. Let $h = l - 2^{n-1}$. Hence, we have $3 \leqslant h \leqslant 2^{n-1} - 1$. It is obvious that the set of vertices $\{\mathbf{x},\mathbf{y},(\mathbf{x})^2,(\mathbf{y})^2\}$ induces a cycle of length 4. By Lemma 3, there exists a Hamiltonian path R of CQ^0_{n-1} joining $\mathbf{x}$ and $(\mathbf{x})^2$. By the inductive hypothesis, there exists a Hamiltonian path S of CQ^1_{n-1} with $S(1) = (\mathbf{y})^2$ and $S(h+1) = \mathbf{y}$. For convenience, path S is written as $\langle (\mathbf{y})^2,S_1,\mathbf{y},S_2,\mathbf{z}\rangle$, where $\mathbf{z}$ is some vertex in $V(CQ^1_{n-1}) - \{(\mathbf{y})^2\}$. It is noticed that $\mathbf{z} = \mathbf{y}$ if $h = 2^{n-1} - 1$. Then, $P = \langle \mathbf{x},R,(\mathbf{x})^2,(\mathbf{y})^2,S_1,\mathbf{y},S_2,\mathbf{z}\rangle$ is a Hamiltonian path of CQ_n with $P(1) = \mathbf{x}$ and $P(l+1) = \mathbf{y}$. Fig. 4(f) shows the illustration. □

Theorem 2. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be any two vertices of* CQ_n, $n \geqslant 4$, *and let* l *be any integer with* $d_{CQ_n}(\mathbf{x},\mathbf{y}) \leqslant l \leqslant 2^{n-1}$ *and* $l \neq d_{CQ_n}(\mathbf{x},\mathbf{y}) + 1$. *There exists a Hamiltonian cycle* C *of* CQ_n *such that* $d_C(\mathbf{x},\mathbf{y}) = l$.

Proof. When $n = 4$, the statement can be verified by brute force with a computer program [31]. Here, we show that there exists a Hamiltonian cycle C of CQ_n, $n \geqslant 5$, such that $d_C(\mathbf{x},\mathbf{y}) = l$. Since $CQ_n = CQ^0_{n-1} \otimes CQ^1_{n-1}$, we assume, without loss of generality, that $\mathbf{x} \in V(CQ^0_{n-1})$. Consider the following three cases.

Case 1. $\mathbf{y} \in V(CQ^0_{n-1})$. In this case, we have to consider the following two subcases.

Subcase 1.1. $d_{CQ_n}(\mathbf{x},\mathbf{y}) \leqslant l \leqslant 2^{n-1} - 1$ and $l \neq d_{CQ_n}(\mathbf{x},\mathbf{y}) + 1$. By letting $\mathbf{a}$ be vertex $(\mathbf{x})^n$ in the proof of Subcase 1.1 in Theorem 1, we can obtain a Hamiltonian cycle $C = \langle \mathbf{x},R_1,\mathbf{y},R_2,\mathbf{z},(\mathbf{z})^n,S,(\mathbf{x})^n,\mathbf{x}\rangle$ of CQ_n such that $d_C(\mathbf{x},\mathbf{y}) = l$.

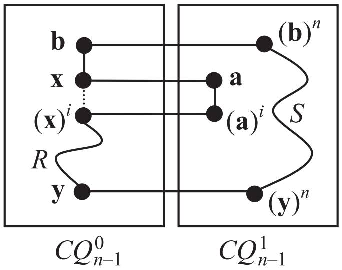

Fig. 5. Subcase 1.2 in the Proof of Theorem 2. (A dashed line or a straight line represents an edge.)

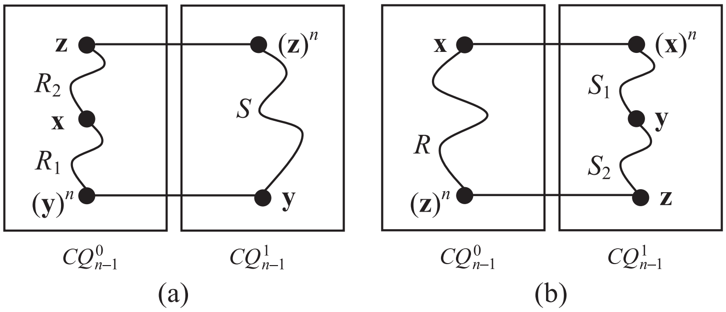

Fig. 6. Case 2 in the Proof of Theorem 2. (A straight line represents an edge.)

Subcase 1.2. $l = 2^{n-1}$. Let $\mathbf{a} = (\mathbf{x})^n$. Obviously, we can choose an even integer i, $1 \leqslant i \leqslant n-1$, such that $(\mathbf{x})^i$ is not identical to $\mathbf{y}$. By Corollary 1, $\{\mathbf{x}, \mathbf{a}, (\mathbf{x})^i, (\mathbf{a})^i\}$ induces a cycle of length 4. Let $\mathbf{b}$ be a neighbor of $\mathbf{x}$ in CQ^0_{n-1} such that $\mathbf{b} \notin \{(\mathbf{x})^1, (\mathbf{x})^i, \mathbf{y}\}$. By Corollary 2, $CQ^0_{n-1} - \{\mathbf{x}, \mathbf{b}\}$ has a Hamiltonian path R joining $(\mathbf{x})^i$ and $\mathbf{y}$. Similarly, $CQ^1_{n-1} - \{\mathbf{a}, (\mathbf{a})^i\}$ has a Hamiltonian path S joining $(\mathbf{y})^n$ and $(\mathbf{b})^n$. Then, $C = \langle \mathbf{x}, \mathbf{a}, (\mathbf{a})^i, (\mathbf{x})^i, R, \mathbf{y}, (\mathbf{y})^n, S, (\mathbf{b})^n, \mathbf{b}, \mathbf{x} \rangle$ is a Hamiltonian cycle of CQ_n with $d_C(\mathbf{x}, \mathbf{y}) = 2^{n-1}$, see Fig. 5 for illustration.

Case 2. $\mathbf{y} \in V(CQ^1_{n-1})$ and $(\mathbf{x}, \mathbf{y}) \notin E(CQ_n)$. By Lemmas 1 and 2, we have $d_{CQ^0_{n-1}}(\mathbf{x}, (\mathbf{y})^n) = d_{CQ_n}(\mathbf{x}, \mathbf{y}) - 1$ or $d_{CQ^1_{n-1}}((\mathbf{x})^n, \mathbf{y}) = d_{CQ_n}(\mathbf{x}, \mathbf{y}) - 1$. For convenience, let $h = l - 1$.

Suppose that $d_{CQ^0_{n-1}}(\mathbf{x}, (\mathbf{y})^n) = d_{CQ_n}(\mathbf{x}, \mathbf{y}) - 1$. Since $d_{CQ^0_{n-1}}(\mathbf{x}, (\mathbf{y})^n) \leqslant h \leqslant 2^{n-1} - 1$ and $h \neq d_{CQ^0_{n-1}}(\mathbf{x}, (\mathbf{y})^n) + 1$, Theorem 1 ensures that there exists a Hamiltonian path R of CQ^0_{n-1} with $R(1) = (\mathbf{y})^n$ and $R(h+1) = \mathbf{x}$. For convenience, path R is written as $\langle (\mathbf{y})^n, R_1, \mathbf{x}, R_2, \mathbf{z} \rangle$, where $\mathbf{z}$ is some vertex in $V(CQ^0_{n-1}) - \{(\mathbf{y})^n\}$. It is noticed that $\mathbf{z} = \mathbf{x}$ if $h = 2^{n-1} - 1$. By Lemma 3, there exists a Hamiltonian path S of CQ^1_{n-1} joining $\mathbf{y}$ to $(\mathbf{z})^n$. Then, $C = \langle \mathbf{x}, rev(R_1), (\mathbf{y})^n, \mathbf{y}, S, (\mathbf{z})^n, \mathbf{z}, rev(R_2), \mathbf{x} \rangle$ is a Hamiltonian cycle of CQ_n such that $d_C(\mathbf{x}, \mathbf{y}) = h + 1 = l$. The illustration of this subcase is shown in Fig. 6(a).

Suppose that $d_{CQ^1_{n-1}}((\mathbf{x})^n, \mathbf{y}) = d_{CQ_n}(\mathbf{x}, \mathbf{y}) - 1$. Since $d_{CQ^1_{n-1}}((\mathbf{x})^n, \mathbf{y}) \leqslant h \leqslant 2^{n-1} - 1$ and $h \neq d_{CQ^1_{n-1}}((\mathbf{x})^n, \mathbf{y}) + 1$, by Theorem 1, there exists a Hamiltonian path S of CQ^1_{n-1} with $S(1) = (\mathbf{x})^n$ and $S(h+1) = \mathbf{y}$. Path S can be written as $\langle (\mathbf{x})^n, S_1, \mathbf{y}, S_2, \mathbf{z} \rangle$, where $\mathbf{z}$ is some vertex in $V(CQ^1_{n-1}) - \{(\mathbf{x})^n\}$. It is noticed that $\mathbf{z} = \mathbf{y}$ if $h = 2^{n-1} - 1$. By Lemma 3, there exists a Hamiltonian path R of CQ^0_{n-1} joining $(\mathbf{z})^n$ to $\mathbf{x}$. Then, $C = \langle \mathbf{x}, (\mathbf{x})^n, S_1, \mathbf{y}, S_2, \mathbf{z}, (\mathbf{z})^n, R, \mathbf{x} \rangle$ is a Hamiltonian cycle of CQ_n such that $d_C(\mathbf{x}, \mathbf{y}) = h + 1 = l$, see Fig. 6(b) for illustration.

Case 3. $\mathbf{y} \in V(CQ^1_{n-1})$ and $(\mathbf{x}, \mathbf{y}) \in E(CQ_n)$. Since $(\mathbf{x}, \mathbf{y}) \in E(CQ_n)$, we have $1 \leqslant l \leqslant 2^{n-1}$ and $l \neq 2$. When $l = 1$, it follows from Lemma 3 that there exists a Hamiltonian path R of CQ_n joining $\mathbf{y}$ to $\mathbf{x}$, and $C = \langle \mathbf{x}, \mathbf{y}, R, \mathbf{x} \rangle$ is our required cycle with $d_c(\mathbf{x}, \mathbf{y}) = 1$. Therefore, we discuss $3 \leqslant l \leqslant 2^{n-1}$. The following three subcases are distinguished.

Subcase 3.1. $l = 3$. As described in the proof of Subcase 3.1 of Theorem 1, we can construct a Hamiltonian cycle $C = \langle \mathbf{x}, (\mathbf{x})^2, (\mathbf{y})^2, \mathbf{y}, R, \mathbf{z}, \mathbf{x} \rangle$ of CQ_n with $d_C(\mathbf{x}, \mathbf{y}) = 3$.

Subcase 3.2. $l = 4$. By using the same argument as that in Subcase 3.2 of Theorem 1, we can construct a Hamiltonian cycle $C = \langle \mathbf{x}, (\mathbf{x})^1, ((\mathbf{y})^1)^2, (\mathbf{y})^1, \mathbf{y}, S, \mathbf{v}, (\mathbf{v})^n, R', \mathbf{u}, \mathbf{x} \rangle$ of CQ_n with $d_C(\mathbf{x}, \mathbf{y}) = 4$.

Subcase 3.3. $5 \leqslant l \leqslant 2^{n-1}$. This subcase is the same as Subcase 3.3 of Theorem 1, and $C = \langle \mathbf{x}, rev(R_1), (\mathbf{x})^2, (\mathbf{y})^2, \mathbf{y}, S, (\mathbf{z})^n, \mathbf{z}, rev(R_2), \mathbf{v}, \mathbf{x} \rangle$ is a Hamiltonian cycle of CQ_n with $d_C(\mathbf{x}, \mathbf{y}) = l$. □

4. Concluding remarks

The class of crossed cubes is a popular variant of the hypercube network for its nice topological properties. However, the crossed cube is not panpositionably Hamiltonian due to the limitation that it contains no cycle of length 3 as a subgraph. Therefore, we are attracted to propose a relaxed version of panpositionable Hamiltonicity for the crossed cube. In this paper, we solve the problem of embedding a Hamiltonian cycle in the crossed cube such that two required vertices can keep a given distance from each other. To be precise, let **x** and **y** be any two distinct vertices of CQ_n. We show that, for any integer l with $d_{CQ_n}(\mathbf{x},\mathbf{y}) \leqslant l \leqslant 2^{n-1}$ and $l \neq d_{CQ_n}(\mathbf{x},\mathbf{y}) + 1$, there exists a Hamiltonian cycle C of CQ_n such that $d_C(\mathbf{x},\mathbf{y}) = l$.

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References

[1] S. Abraham, K. Padmanabhan, The twisted cube topology for multiprocessors: a study in network asymmetry, J. Parallel Distrib. Comput. 13 (1991) 104–110.
[2] Y. Alavi, J.E. Williamson, Panconnected graphs, Stud. Sci. Math. Hung. 10 (1975) 19–22.
[3] J.A. Bondy, Pancyclic graphs, J. Comb. Theory Ser. B 11 (1971) 80–84.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, London, 2008.
[5] H.-C. Chan, J.-M. Chang, Y.-L. Wang, S.-J. Horng, Geodesic-pancyclicity and fault-tolerant panconnectivity of augmented cubes, Appl. Math. Comput. 207 (2009) 333–339.
[6] C.-P. Chang, T.-Y. Sung, L.-H. Hsu, Edge congestion and topological properties of crossed cubes, IEEE Trans. Parallel Distrib. Syst. 11 (1) (2000) 64–80.
[7] S.A. Choudum, V. Sunitha, Augmented cubes, Networks 40 (2002) 71–84.
[8] K. Efe, A variation on the hypercube with lower diameter, IEEE Trans. Comput. 40 (11) (1991) 1312–1316.
[9] K. Efe, The crossed cube architecture for parallel computing, IEEE Trans. Parallel Distrib. Syst. 3 (5) (1992) 513–524.
[10] K. Efe, P.K. Blackwell, W. Slough, T. Shiau, Topological properties of the crossed cube architecture, Parallel Comput. 20 (1994) 1763–1775.
[11] J. Fan, X. Lin, X. Jia, Optimal path embeddings in crossed cubes, IEEE Trans. Parallel Distrib. Syst. 16 (2) (2004) 1190–1200.
[12] J. Fan, X. Lin, X. Jia, Node-pancyclicity and edge-pancyclicity of crossed cubes, Inf. Process. Lett. 93 (2005) 133–138.
[13] J. Fan, X. Jia, X. Lin, Complete path embeddings in crossed cubes, Inf. Sci. 176 (2006) 3332–3346.
[14] L.-H. Hsu, C.-K. Lin, Graph Theory and Interconnection Networks, CRC Press, 2008.
[15] W.-T. Huang, Y.-C. Chuang, J.J.M. Tan, L.-H. Hsu, On the fault-tolerant hamiltonicity of faulty crossed cubes, IEICE Trans. Fundam. E85-A (6) (2002) 1359–1370.
[16] S.-S. Kao, C.-K. Lin, H.-M. Huang, L.-H. Hsu, Panpositionable Hamiltonian graphs, ARS Comb. 81 (2006) 209–223.
[17] T.-L. Kueng, C.-K. Lin, T. Liang, J.J.M. Tan, L.-H. Hsu, Embedding paths of variable lengths into hypercubes with conditional link-faults, Parallel Comput. 35 (2009) 441–454.
[18] P. Kulasinghe, S. Bettayeb, Embedding binary tree into crossed cubes, IEEE Trans. Comput. 44 (7) (1995) 923–929.
[19] P. Kulasinghe, Connectivity of the crossed cube, Inf. Process. Lett. 61 (1997) 221–226.
[20] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays Trees Hypercubes, Morgan Kaufman, San Mateo, 1992.
[21] M. Ma, G. Liu, J.-M. Xu, Panconnectivity and edge-fault-tolerant pancyclicity of augmented cubes, Parallel Comput. 33 (2007) 36–42.
[22] J.-H. Park, H.-C. Kim, H.-S. Lim, Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements, IEEE Trans. Parallel Distrib. Syst. 17 (3) (2006) 227–240.
[23] Y. Saad, M.H. Shultz, Topological properties of hypercubes, IEEE Trans. Comput. 37 (1988) 867–872.
[24] Y.-H. Teng, J.J.M. Tan, L.-H. Hsu, Panpositionable Hamiltonicity of the alternating group graphs, Networks 50 (2007) 146–156.
[25] Y.-H. Teng, J.J.M. Tan, L.-H. Hsu, Panpositionable Hamiltonicity and panconnectivity of the arrangement graphs, Appl. Math. Comput. 19 (2008) 414–432.
[26] J.-M. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.
[27] M.-C. Yang, T.-K. Li, J.J.M. Tan, L.-H. Hsu, Fault-tolerant cycle embedding of crossed cubes, Inf. Process. Lett. 88 (2003) 149–154.
[28] X. Yang, D.J. Evans, G.M. Megson, The locally twisted cubes, Int. J. Comput. Math. 82 (4) (2005) 401–413.
[29] <http://www.cs.nctu.edu.tw/~tlkueng/DATA/AMC16096/CQ4Lemma6.txt>.
[30] <http://www.cs.nctu.edu.tw/~tlkueng/DATA/AMC16096/CQ4Theorem1.txt>.
[31] <http://www.cs.nctu.edu.tw/~tlkueng/DATA/AMC16096/CQ4Theorem2.txt>.